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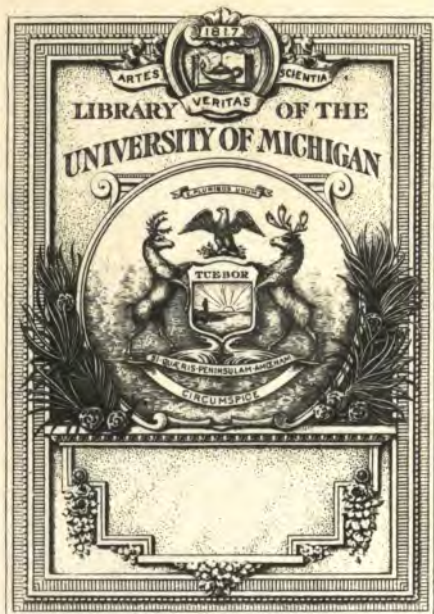
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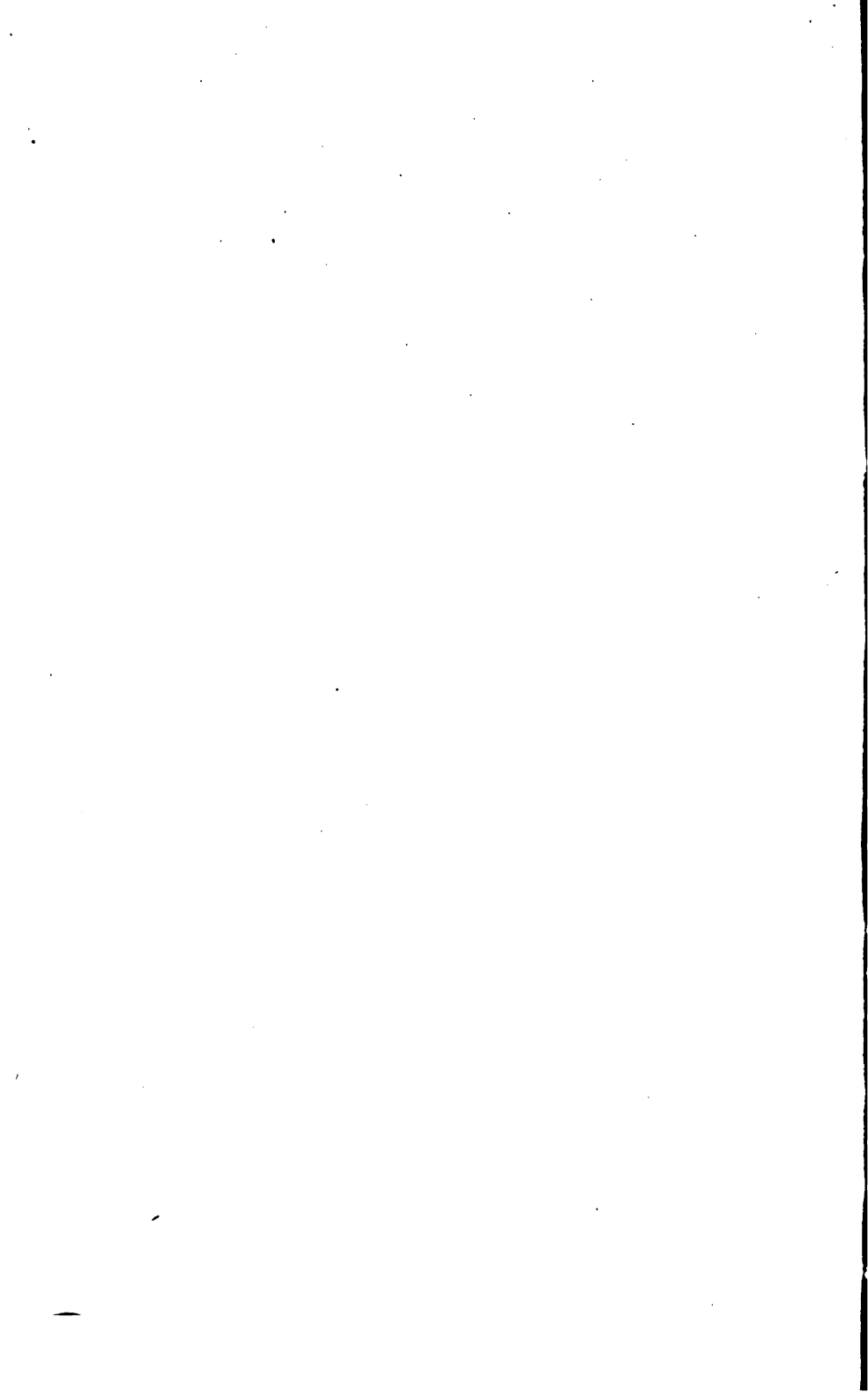
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THE
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A JOURNAL DEVOTED TO LITERATURE, SCIENCE, ART  
AND THE ADVANCEMENT OF THE TEACHING  
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—♦♦♦—
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OF

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DURING THE YEAR 1890-91.

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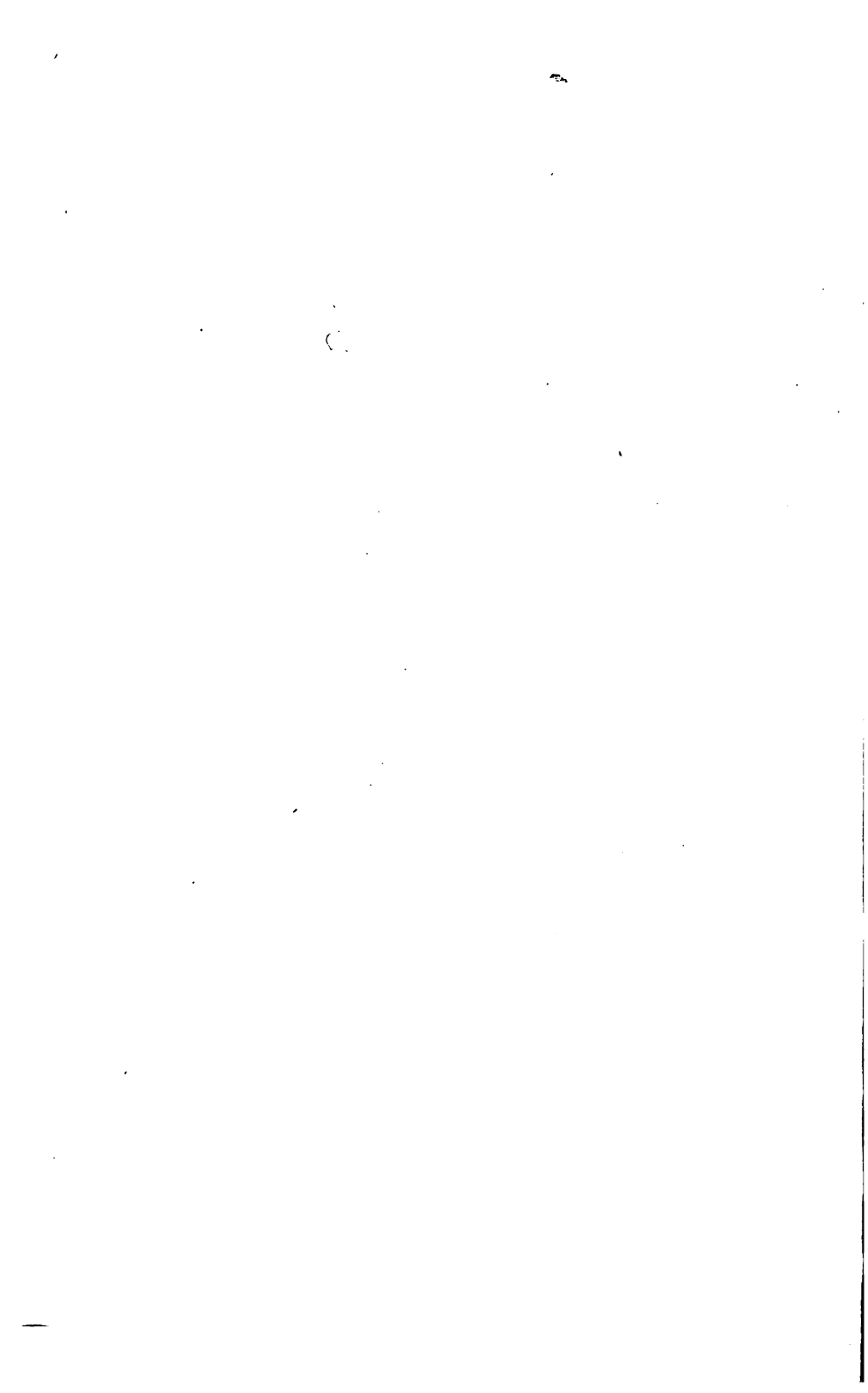
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PREFACE.

Some of the papers read before the Society during the past year being deemed of sufficient excellence to warrant their publication, a committee was appointed for that purpose. That committee has now performed its duty, it hopes acceptably. Should this venture prove a success the Society may, next year, publish a periodical in which the papers as they are read will appear from time to time. This publication should receive the support of the teaching profession and graduates in Mathematics throughout the Province.



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POETIC INTERPRETATION IN MATHEMATICS.

*Delivered by Professor Alfred Baker, M.A., before the
Mathematical and Physical Society of the University
of Toronto.*

GENTLEMEN,—I was somewhat at a loss to determine what title I ought to prefix to my address. My general object is to shew that our subject appeals to the imagination, to our sense of beauty, to the affections, to the emotions; and that, therefore, in some of its developments it comes within the regions haunted by the Muses. The question whether the advance of science will destroy the poetic feeling is an old one, which I think we have abundant reason to answer in the negative. As soon will it destroy the religious feeling. The effect of scientific study on the individual is various, and dependent on the strength of mind and strength of character of the individual. The study of science cannot *create* imagination where nature has not supplied it; nor, for the matter of that, can the study of general literature. The study of general literature may create a pseudo article—may breed an affectation—which, however, never deceives the discerning mind.

Science will not extinguish poetic feeling if it is cultivated in a proper soil. It may modify its character, but by extending knowledge will enlarge its domain. The emotions which a great scientific discovery awakens may be as genuinely poetic as those roused by any well-told tale of the imagination. The former usually are to be found at a greater spiritual altitude than the latter. Thoughts, to be utilized by the poet, must be easy and familiar to him. To appeal to the public, they must be easy and familiar to them. The world has not yet assimilated the scientific truth that recently has been conquered, and it is not yet material for the artist in his appeals to

the public. But limitations that hold with respect to the general public are not binding on us. To have an affection for our subject we must see beauty in it. The æsthetic faculty must be aroused, and the imagination awakened.

My discourse will appear somewhat disjointed. I have selected, almost at random, examples that seem best suited to illustrate my position.

The intellectual powers of man have enabled him to create for himself nothing more exquisite in its structure, or more refined in its applications than the infinitesimal calculus. "A thing of beauty is a joy forever." Conversely, I submit, what is a joy forever must have beauty for its essential quality. I never tire of the calculus. Surely it is its beauty that allures me. If we turn to nature we find all things in a state of continual change, and this change is not abrupt, but gradual. At times it is slow and silent, as in the unfolding of vegetable life; at times it is rapid, as when a body falls to the earth. In all cases the change comes by additions, in the limit, so minute as utterly to escape our powers of observation. Consider the case of a falling body. At any moment its velocity is definite, but yet not even for the millionth part of a second does it remain the same. Long before the mind can have conceived the quantity that may represent the velocity, it has eluded it and become something else. Or again,—think of a point by its motion generating a curve. Not for the smallest part of a moment does its direction remain the same, and not for the twinkling of an eye does the addition to the area of the curve remain the same. Conceive then the enormous difficulty of bringing the definiteness of mathematics to bear on things so utterly illusory. It seems like trying to overtake the lightning. And yet such are the questions the infinitesimal calculus proposes to itself

And now I shall attempt to give you an elementary lesson in the calculus—a very elementary lesson indeed. [Prof. Baker here gave illustrations in differentiation and integra-

tion in their simplest possible forms.] Such are illustrations of the processes called differentiation and integration.

Bishop Berkeley, who was disposed to scoff at the great invention of Newton and Leibnitz, spoke of differentials as the "ghosts of departed quantities." The figure is in some respects an admirable one; and accepting it we can claim that the mathematician has more than the magic power to summon spirits back to earth,—he actually restores to life itself the beings with which his world is peopled. The microscope reveals to us an existence of which our senses could take no cognizance; but, in a way, the calculus steps in when the microscope has ceased to penetrate, and when the mind refuses to conceive the further divisibility of space this wonderful method continues indefinitely its faultless work. Our intelligence is almost staggered in its attempts to realize star-distances; yet the calculus makes thinkable spaces compared with which star-distances are vanishing quantities. By it, as in the case of singular solutions of differential equations, and indeed in other branches of mathematics when solving problems, we are often led to truths of the existence of which we never dreamt when proposing such problems; and thus we will be pardoned for fancifully claiming that while man constantly erects monuments that survive him, he has here created a something whose intelligence, in a sense, surpasses his own. It may be urged by some one,—
 "Such results are due to the necessary and mechanical operation of the logical laws on which the science is founded." With such an one I have no dispute—

" A primrose by a river's brim
 A yellow primrose is to him,
 And it is nothing more."

The ideal perfection of geometrical figures, their marvellous symmetry; the purity, if I may so express it, of mathematical law, and the entire freedom it enjoys from disturbing influences, have lead minds disposed to mysticism to

attribute to mathematical truth a certain elevated symbolism. The system I look upon as little better than a conceit; but yet it cannot consistently be assailed by those whose poetic interpretation of nature consists only in detecting analogies between it and the spiritual or moral universe. Let me illustrate this symbolism by a selection from Kingsley's "Hypatia." Hypatia, who devoted herself to philosophy and science, had written a treatise on conic sections. She and her father with his pupil Philammon are walking in the Museum Gardens of Alexandria; and the conversation thus proceeds:

"Well," asked the old man with an encouraging smile, "and how does our pupil like his new—"

"You mean my conic sections, father? It is hardly fair to expect an unbiased answer in my presence."

"Why so?" said Philammon. "Why should I not tell you as well as all the world, the fresh and wonderful field of thought which they have opened to me in a few short hours?"

"What then?" asked Hypatia, smiling, as if she knew what the answer would be. "In what does my commentary differ from the original text of Apollonius, on which I have so faithfully based it?"

"O, as much as a living body differs from a dead one. Instead of mere disquisitions on the properties of lines and curves, I found a world of poetry and theology. Every dull mathematical formula seems transfigured, as if by a miracle, into some deep and noble principle of the unseen world."

"And do you think that he of Perga (Apollonius) did not see as much? or that we can pretend to surpass in depth of insight the sages of the elder world? Be sure that they, like the poets, meant only spiritual things, even when they seem to talk only of physical ones, and concealed heaven under an earthly garb only to hide it from the eyes of the profane; while we, in these degenerate days, must interpret and display each detail to the dull ears of men."

"Do you think, my young friend," asked Theon (the father), "that mathematics can be valuable to the philosopher otherwise than as vehicles of spiritual truth? Are we to study numbers merely that we may be able to keep accounts; or as Pythagoras did, in order to deduce from their laws the ideas by which the universe, man, Divinity itself, consists?"

"That seems to me certainly to be the nobler purpose."

"Or conic sections, that we may know better how to construct machinery; or rather to devise from them symbols of the relations of Deity to its various emanations?"

* * * *

"And what is the meaning of the circle?" asked Philammon.

"It may have infinite meanings, like every other natural phenomenon; and deeper meanings in proportion to the exaltation of the soul which beholds it. But, consider, is it not, as the one perfect figure, the very symbol of the totality of the spiritual world; which, like it, is invisible, except at its circumference, where it is limited by the dead gross phenomena of sensuous matter? And even, as the circle takes its origin from one centre, itself unseen—a point, as Euclid defines it, whereof neither parts nor magnitude can be predicated,—does not the world of spirits revolve round our abysmal being, unseen and undefinable,—in itself, as I have so often preached, nothing, for it is conceivable only by the negation of all properties, even of those of reason, virtue, force; and yet, like the centre of the circle, the cause of all other existences?"

"I see," said Philammon;

* * * *

"Let that be enough for the present. Hereafter you may be—I fancy that I know you well enough to prophesy that you will be—able to recognize in the equilateral triangle inscribed within the circle, and touching it only with its angles, the three supra-sensual principles of existence,

which are contained in Deity as it manifests itself in that physical universe, coinciding with its utmost limits, and yet, like it, dependent on that unseen central One, which none dare name."

"Ah!" said poor Philammon, blushing scarlet at the sense of his own dulness, "I am indeed not worthy to have such wisdom wasted upon my imperfect apprehension. But, if I may dare to ask: does not Apollonius regard the circle, like all other curves, as not depending primarily on its own centre for its existence, but as generated by the section of any cone by a plane at right angles to its axis?"

"But must we not draw, or at least conceive, a circle, in order to produce that cone? And is not the axis of that cone determined by the centre of that circle?"

Philammon stood rebuked.

"Do not be ashamed; you have only unwittingly laid open another and perhaps as deep a symbol. Can you guess what it is?"

Philammon puzzled in vain.

"Does it not shew you this? That, as every conceivable right section of the cone discloses the circle, so in all that is fair and symmetric you will discover Deity, if you but analyze it in a right and symmetric direction?"

"Beautiful!" said Philammon; while the old man added,

"And does it not shew us, too, how the one perfect and original philosophy may be discovered in all great writers if we have but that scientific knowledge which will enable us to extract it?"

Will you allow me, gentlemen, to restore, by conjecture, what might have formed a page of Hypatia's lost treatise on the conics, as follows:—

Even as a right section of the cone reveals the circle, with its perfect symmetry, which symbolizes the totality of the spiritual universe, and centre, Divinity; so an oblique section of the cone reveals the ellipse of wonderful

but imperfect symmetry. The divinity which the centre of the circle symbolized has become divided, but its constancy and unchangeableness is still represented by the constancy of the distances to its parts,—the sum of the focal distances. The directrix may suggest the written law, and the law of the conic symbolizes the constancy in the spiritual world of the relation of the Deity to law; and, even as, when the ellipse becomes a circle, the directrix recedes indefinitely, so, in the true view of life, when the perfection and absolute symmetry of Deity stand forth, the written law recedes indefinitely, and we see the spiritual universe revolving in absolute dependence on the great first Cause.

I might even continue the system into mathematical truths with which, possibly, Hypatia had no acquaintance, thus:—

A curve created in accordance with a law that we conceive in finite regions, but which yet has branches receding into infinite space, which still observe the law of the curve's creation, typifies well the immortality of man. The asymptote—a straight line—may be considered to symbolize divine perfection; and just as the curve continually gets nearer and approximates to the asymptote, but yet, through infinite space, never attains it; so the soul of man in a future state may continually approach divine perfection, but yet, in the endless roll of the eternal ages never absolutely attain it.

The curve at every point of its length coincides with a circle, called the circle of curvature, whose radius grows as the curve recedes into space. How forcibly this reminds us of a constant contact in a future life with a growing perfection and inspiration, adapting itself constantly to the capacities of our being! But some one may say: "Are you not inconsistent—formerly you took a straight line to represent perfection, and now you use a circle?" Not at all. A straight line is a circle of infinite radius. The-

limited circle suggests a limited perfection ; the straight line, an infinite perfection that is all-embracing.

Rousseau, in his "Confessions" tells us that when Father Castel was advising him how to proceed in order to secure the introduction of his new system of musical notation, Castel, who must have been a mathematician, said, "Nothing is done in Paris without the women. They are the curves of which the wise are the asymptotes : they incessantly approach each other, but never touch."

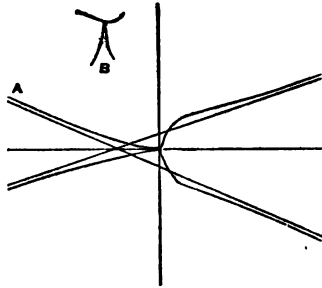
The application of the mathematical truth that an infinite quantity is not increased by the addition of a finite one, nor diminished by taking it away, I leave you to make for yourselves.

The old Greeks, with more imagination than ourselves, had the muse Urania presiding over astronomy ; and in what poetic aspects must mathematics have presented itself to the Pythagoreans and Platonists, who thought the great secret of the universe was to be found in number and in form ! Only poets could have had such an inspiration, for it was an anticipation of some of the grandest discoveries of modern science, as, for example, in the theory of light and colour. What they divined, we prove (how the word "prove" seems to rob truth of its poetry)—we prove—and see how nature hymns the well-known numbers through innumerable variations.

Few studies at once cultivate and discipline the imagination better than geometry. It was for this reason, as well as for its value as a basis of scientific education, that Plato inscribed over the entrance to his school, "Let none ignorant of geometry enter my door." Let me illustrate by an example in curve tracing how the imagination is exercised :—

From the equation to a curve we learn that it is symmetrical with respect to a line (Ox) ; that it meets at infinity two lines that intersect on this line ; that to the right of this point the curve has the form B ; that at remote

distances the curved lines without the lines that reach it at infinity; and from all the infinite forms a curve may take we have to imagine one that fulfils these conditions. The curve will be found to be this:—



It will be seen that the exercise of imagination is precisely the same as when from observed phenomena a generalization is made that constitutes a great natural law.

Compactness combined with clearness of expression is a beauty in any language that possesses it. Coleridge in his "Greek Classic Poets" speaks of Greek as a language of indefatigable strength, with the distinctness of nature herself, with words like pictures, and as not being compressed to its closest by Thucydides. How attractive that same sententiousness, that power of saying most in fewest words, is in any author or in any speaker! How we feel the beauty as we read the essays of Bacon! Yet compared with the language of mathematics the strength of utterance of the greatest author or speaker is "as moonlight unto sunlight, and as water unto wine."

If we consider the general equation of the second degree,

$$ax^2 + by^2 + 2cxy + 2dx + 2ey + f = 0$$

we find in it a world of meaning. Volumes and volumes can be written, yet every truth in them is locked up in the silent signs and symbols of this single line. To the eye of the mathematician it teems with life. There are to be found the ellipse with its wonderful symmetry and its suggestion of planetary paths; the hyperbola with its far reaching infinite branches; and the parabola with its reminders of cometary orbits. We have only to look deep enough into it and we see the law of gravitation. A proper analysis of it reveals the foci and the directrices, the

constancy of the sum or difference of the focal distances, the foci as the poles of the directrices, the laws of intersecting tangents, the surprises of reciprocation, and the myriad other wonderful properties of these curves that for two thousand years have attracted the attention of the world of science.

Again,—in the Lunar Theory we have the following expression for the longitude (θ) of the moon at any time t ,—

$$\begin{aligned} \theta = & pt \\ & + 2e \sin (cpt - a) + \frac{5}{4} e^2 \sin 2 (cpt - a), \text{ elliptic } \pm ty \\ & + \frac{15}{4} me \sin \left\{ (2 - 2m - c)pt - 2\beta + a \right\}, \text{ evection} \\ & + \frac{11}{8} m^2 \sin \left\{ (2 - 2m)pt - 2\beta \right\}, \text{ variation} \\ & - 3me \sin (mpt + \beta - \zeta), \text{ annual} = n, \end{aligned}$$

the succeeding terms being corrections on the first (pt). The *elliptic inequality* tells us that the apse of the moon's orbit is not stationary, but has a progressive motion in longitude equal to $(1 - c)p$.

The *evection* tells us that this progressive motion of the apse is variable; and also that the eccentricity of the moon's orbit is subject to change, that the great curve vibrates as it were. It tells us that this change in the shape of the orbit is periodic, and not one that might result in the destruction of the system. What good news, what a genuine gospel a mathematical term may bring!

The *variation* shews where the moon's true place is before and where after its mean place.

The *annual equation* reveals to us that the angular velocity of the moon is least when the sun is nearest the earth and greatest when most remote.

Is there anything that you can name that has so far-reaching a meaning as these terms! The great orb of night as she rolls through space has her story told for ages in this distich. And what a vastness of human labour in the pro-

gress of science do these symbols suggest! As we thus consider, a certain grandeur seems to clothe them, hardly surpassed by the queen of night herself as she sweeps the sky. Fancy almost conceives the terms themselves as constraining the planet in her orbit by the mighty power of truth; and in their pure intellectuality there rests a beauty that equals that of the moonlight itself as it slumbers on a summer sea.

I know of nothing in the romance of science comparable with the discovery of the planet Neptune: there is in it so much of cosmic grandeur, and so vast a thought of human genius and human toil. How long and weary the road to be travelled, that the primitive savage, gazing with ignorant awe on the starry heavens, might evolve into Adams, or Le Verrier, in his quiet study, with his calculations spread out before him, and saying, as he brings them to a close, "A planet on which human eye has never looked, far off there on the confines of the solar system, whose presence is only revealed by its subtle influence on another planet almost as remote, still another wanderer circling round the sun will, on the night of September 24th, 1846, be found in celestial longitude 326° ."

Let me explain: The planets, proceeding outwards from the sun, are Mercury, Venus, Earth, Mars, Asteroids, Jupiter, Saturn, Uranus, and Neptune. The distance of Uranus from the sun is 1,781 millions of miles; that of Neptune 2,791 millions. Sir Wm. Herschel, in 1781, had discovered Uranus by telescopic observation. In 1821, Bouvard, an eminent French astronomer, prepared tables of Uranus, representing its motions in the heavens. Now let me say that with tables such as these, and being given the masses, in this case of the Sun, Saturn, and Jupiter and the law of gravitation, it was possible to calculate the position that at any future time Uranus ought to occupy. These calculations were made, but the telescope failed to reveal the planet in the place to which theory consigned

it. What was the cause? Was there a resisting medium interfering with its motions? Was it that the law of the sun's attraction as it penetrated those almost infinite depths broke down? Or was there a hitherto unseen and unknown planet beyond Uranus whose influence produced those disturbances in its motion? Almost simultaneously Le Verrier in Paris and Adams in Cambridge devoted themselves to the question. The problem was of extreme difficulty. When the mass and position of a body are known it is not hard to determine the disturbances it will produce in another body; but the problem these two mathematicians attacked was the reverse of this, viz., given the disturbances to find the mass and position of the distributing body. For upwards of a year they laboured on. Adams sent his results to Airy; Le Verrier published his, sending his third memoir to Encke, of Berlin, with the request that he search telescopically for the planet. At the meeting of the British Association in September, 1846, while the matter was still pending, Sir John Herschel said, "The past year has given us the new planet *Astræa* (one of the asteroids.) It has done more—it has given us the probable prospect of the discovery of another. We see it as Columbus saw America from the shores of Spain. Its movements have been felt, trembling along the far-reaching line of our analysis." The search was successful at the Berlin observatory. Galle, Encke's assistant, directed the great telescope to that region of the heavens where it was predicted the planet would appear, and called out the visible stars, while D'Arrest, a young student, checked them by the star map. After a while Galle saw what seemed to be a star of the 8th magnitude, which was not laid down in the map. Further observation proved that this was the sought-for planet. The picture that here rises in our minds of Adams and Le Verrier, from the quiet of their lonely studies uplifted and borne onward through illimitable distance by the strong winds of mathematical truth until

they stand on another world,—the picture, I say, is sublime. The greatest of German writers has presented to us a scholar tired of life and weary of his knowledge, summoning to his assistance from the spirit world an uncanny helper. When divested of the garb with which genius clothes all it conceives, has this creation of Goethe anything of the poetry and beauty and meaning that suggest themselves to us as we think of Adams and LeVerrier summoning up their *Neptune* from the depths of the ocean of infinite space?

It is difficult to over-estimate the debt that the scientific faculty in general owes to mathematics in the past and also in the present. I mean that the study of mathematics possibly first suggested to mankind the existence, amid the confusion of natural phenomena, of fixed law. In nature a law seldom operates without interference, and in consequence first observations reveal what apparently is hopeless and inextricable confusion. Atmospheric pressure prevents projectiles from describing accurate parabolas. The same influence retards a falling body so that the velocity acquired is not proportional to the time. The planets move constantly in the same direction, but observation of them from our moving earth makes them appear sometimes progressive, sometimes retrograde, and at times to be motionless. But in mathematics we have the unimpeded action of law. Thus in geometry the ancients were acquainted with the fact that as a line swung about a point and intersected a circle, though the length of the segments were varying their product was always the same; that whatever the sides of a right-angled triangle the sum of their squares was always equal to the square on the hypotenuse; and that similarly constancy existed amid variety in other cases. This knowledge, it is more than possible, first suggested to mankind that amid all the confusion of natural phenomena fixed laws were operating capable of definite expression in simple form. And so it came that Hippar-

thus sought to explain the irregularities in the moon's motion by supposing it to move on a circle of which the earth was not exactly the centre. In general the physicists like Archimedes were also mathematicians. But what I wish here more particularly to say is that these laws in their perfection and simplicity appeal to the æsthetic faculty. Beauty may be defined as that proportion of parts, that harmony, sometimes that symmetry, which produces an agreeable emotion or feeling. The appeal is at times to the senses, at others directly to the intellect; and the pleasurable sensation is heightened by the detection of analogies and congruences, and the discovery of a power to sum these observations in simple form. Our subject teems with beauty viewed from this standpoint, especially in geometry. But take an illustration of the simplest kind borrowed from abstract mathematics.

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \dots\dots\dots$$

The harmony and symmetry that pervades the series appeals at once in however elementary a way to our sense of beauty. The imagination is allured in the contemplation of the series flowing on forever, and the element of mysticism is supplied in the dimness that accompanies the ultimate evanescence of the terms, while as correspondent to the repose the mind seeks in everything artistic, we have the equivalence of the series in the sententious form e^x .

In geometry some of the higher curves are marvels of grace and symmetry, and the determination of their forms from their expression in analytic language—called their equations—awakens our liveliest æsthetic sense. Even the comparatively simple and elementary ellipse illustrates this. Has it ever occurred to you that the ellipse is as much a product of nature as the enduring mountains or the changing ocean? The huge planets were describing it with all its wonderful properties, in illimitable space countless ages ago, before man appeared on this globe. The harmo-

nies that exist in this curve seem to have been dimly felt by some of our greatest poets. Hence, let me claim, amongst the ancients we have the "music of the spheres." So also Goethe says,—

"Still quiring as in ancient time
With brother spheres in rival song.
The sun with thunder-march sublime
Moves his predestined course along."

"Quiring" and "song" as if, it may be, the properties of the curve were the notes of a march of triumph hymned forth by the day-god in his progress through the heavens; and "predestined," as if the great ruler of the solar system were bound by the inflexible law of the ellipse, as by a fate.

And our own great dramatic poet says,—

"Look how the floor of heaven
Is thick inlaid with patines of bright gold.
There's not the smallest orb which thou behold'st
But in his motion like an angel sings,
Still quiring to the young-eyed cherubins;
Such harmony is in immortal souls;
But whilst this muddy vesture of decay
Doth grossly close it in, we cannot hear it."

ON CERTAIN DEDUCTIONS FROM THE THEOREM OF DR. GRAVES.

By A. T. DeLury, B.A.

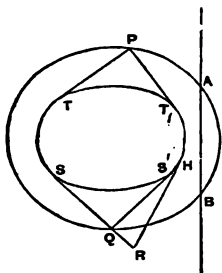


Fig. 1.

The theorem of Dr. Graves referred to is the following: From any pt. P on the exterior of two confocal ellipses (Fig. 1) let tangents PT and PT' be drawn to the interior ellipse; then the difference between the sum of the tangents $PT + PT'$ and the arc TT' is constant for all positions of P on the exterior ellipse. Thus if Q be another pt. on the exterior ellipse and QS QS' tangents to the interior, $PT + PT' - \widehat{TT'} = QS + QS' - \widehat{SS'}$, or the difference between the arcs TT' and SS' is a st. line. Two arcs of the same ellipse whose difference is thus rectifiable are called *similar arcs*, and an angle as TPT' is said to be *circumscribed* to the arc TT' .

In the following paper will be given certain propositions that depend on this theorem for their proof. They were first given by Chasles in the volume of *Comptes Rendus* for the year 1843. The theorems are all stated there, and this paper differs from a translation of the article referred to only in having, in some cases, the connection of the theorems with the theorem of Graves pointed out more explicitly.

It is assumed that no portion of the arc of an ellipse is rectifiable; this may easily be arrived at analytically, and is necessary to the proof of Prop. I.

Prop. I.—Two similar arcs of an ellipse have always the summits of their circumscribed angles situated on a second conic section, confocal with the first. For let TT' & SH

(Fig. 1), be two similar arcs and P and R , the summits of the circumscribed π . Through P describe the confocal ellipse (the solution is singular). This will pass through R ; for if not let it cut RS in Q . From Q draw the tangent QS' . Then by Graves' Theorem SS' and TT' differ by a certain st. line: so do SH and TT' $\therefore$ SH and SS' differ by a st. line. i.e. $S'H$ is rectifiable which is absurd $\therefore$ etc.

The difference between TT' and SS' is equal to the difference between the sum of the sides of the angle circumscribed to TT' and the sum of the sides of the angle circumscribed to SS' .

Also to find an arc starting at S similar to TT' , describe through P the confocal ellipse, and from where it cuts the tangent at S , draw the other tangent through this point. This gives the extremity of the similar arc.

By repeating this a multiple of an arc—neglecting differences that can be rectified—can be obtained. The subdivision of an arc into parts whose differences is rectifiable may be reduced to the solution of an algebraic $=^n$. For example to divide an arc of an ellipse, the eccentric π of whose extremities are a and β into two such parts. Let $=^n$ of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$: Let ϕ give the pt. required the tangents at (a) and (ϕ) intersect in the point whose co-ordinates are

$$\left(a. \frac{\sin \phi - \sin a}{\sin (\phi - a)} \quad b. \frac{\cos a - \cos \phi}{\sin (\phi - a)} \right) \dots\dots\dots A.$$

and those at β and ϕ in the pt. gn. by

$$\left(a. \frac{\sin \phi - \sin \beta}{\sin (\phi - \beta)} \quad b. \frac{\cos \beta - \cos \phi}{\sin (\phi - \beta)} \right) \dots\dots\dots B.$$

Let $\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} - 1 = 0$ be the $=^n$ of the confocal passing through (A) and (B) . The condition that this passes through (A) and (B) gives two $=^n$ s, from which λ can be eliminated leaving an algebraic $=^n$ for $\sin \phi$ or $\cos \phi$.

The way of proceeding for a division into three or more parts follows readily.

All similar arcs, then on an ellipse, have the summits of their circumscribed angles on a confocal conic: therefore, their chords all touch an ellipse, the reciprocal polar of the second with respect to the first.

These properties enable us, having given an arc of an ellipse to determine a similar arc satisfying certain conditions. For example:—

(a) That its circumscribed angle should have its summit on a given straight line.

Let TT' (Fig. 1) be given arc, P the summit of the circumscribed angle, and AB the given line. Through P pass the confocal ellipse, cutting the given line in A and B . From either A or B draw the tangents to the original ellipse intercepting similar arcs having the summits of their circumscribed angles in the given line.

(b) That its circumscribed angle should have its summit on a given curve. The construction is similar.

(c) That the circumscribed angle may be of given value.

Construct the locus of points such that the tangents from them to the given ellipse may include an $\angle$ equal to the given angle, and then proceed as in (a) or (b). The locus referred to can be constructed readily analytically.

(d) That the chord subtended by the arc may pass through a given point.

As before, draw the confocal ellipse through P . Draw its reciprocal polar with respect to the original ellipse, and to this last, draw the tangents from the given point. These tangents will intercept on the original ellipse, arcs similar to TT' , the proof being completed by the *ad absurdum* method.

A great many other conditions might be taken but these will suffice to shew the method.

Let TS and $T'S'$ be similar arcs. Then by *Prop. II.* a $\odot$ may be inscribed to the figure $PLQM$ (Fig. 3). Let S approach T' and coincide with it and this $\odot$ will become a $\odot$ touching PK and $\therefore$ touching the ellipse in T'

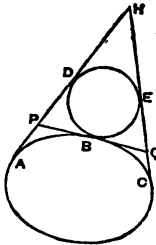


Fig. 4.

Then (Fig. 4) the difference between the similar arcs AB , BC , is the difference between the sum $PA + PB$ and the sum $QB + QC$. But $PB = PD$ and $BQ = QE$ and $HD = HE$. $\therefore$ This difference is the same as the difference between HA and HC . $\therefore$ etc.

By means of McCullagh's Theorem (Williamson Int. Calc. p. 236) it at once appears that the confocal hyperbola through H passes through B , and this gives a means of dividing a given arc into two parts, of which the difference is rectifiable.

Prop. IV.—When two similar arcs have a common extremity, in the angle formed by the tangents at their free extremities, can be inscribed a $\odot$ which will touch the ellipse at their common extremity.

This has been already shewn. Thus when a $\odot$ is tangent to an ellipse, the two common tangents determine two similar arcs.

Prop. V. If an arc is divided into m similar arcs the points of division are such that the tangents to the curves at these points form the portion of a polygon of $M + L$ sides, which has the minimum perimeter among all the portions of the polygon of the same number of sides circumscribed to the given arc.

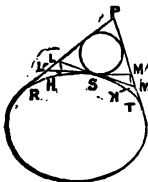


Fig. 5.

Thus for a division into two parts: the arc RT (Fig. 5) is divided at S in the way mentioned. Then $RL + LM + MT$ is to be a minimum. A $\odot$ will touch PL and PM and the ellipse at S . Let LM undergo an infinitesimal displacement, still remaining a tangent and $\therefore$ passing through a pt. in

LM infinitesimally near to S , or through it, if we are considering finite lengths.

Draw $LH \perp$ to $L'M'$ and $M'K \perp$ to LM

Let angle $LSL' = d\theta$

$$\therefore LH = LS \cdot d\theta$$

$$\therefore LL' - L'H = LS \cdot d\theta (-\cot L + \operatorname{cosec} L)$$

$$= LS \cdot d\theta \cdot \left(\frac{1 - \cos L}{\sin L} \right)$$

$$= LS \cdot d\theta \cdot \tan \frac{L}{2}$$

$$= d\theta \cdot \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \text{ in the ordinary notation.}$$

$$= \therefore MM' - KM, \text{ by symmetry}$$

$$\therefore RL + LM + MT = RL' + L'M' + M'T.$$

$\therefore$ An infinitesimal displacement leaves the perimeter unchanged—the condition for a maximum or minimum. That it is a minimum appears from considering the case where LM moves nearly to coincidence with RP .

If there were m arcs, a change in the position—infinitesimal change—of one of the sides would not affect the perimeter. $\therefore$ By the principles of partial differentiation the total change for an infinitely small displacement of *all* the sides is zero, and as before the perimeter is a minimum.

Prop. VI. The same portion of the polygon has its vertices on a second ellipse confocal to the given ellipse, and has with reference to that curve a maximum perimeter.

The first part is evident from Graves' Theorem.

Let L, M, N (Fig. 6) be vertices of such a polygon. Then since the external ellipse is confocal to the inner the lines ML and MN are equally inclined to the curve at M . Let M be displaced infinitesimally to M' . Then if $\perp$ s be drawn from M' and M to LM and NM' it can be easily shewn from this last fact that $LM' + M'N' = LM + MN$. $\therefore$ the change in the peri-

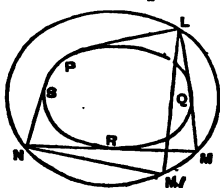


Fig. 6.

meter due to a small displacement of M is zero and similarly for the other vertices and $\therefore$ as before by the fundamental theorem of partial differentiation a small displacement of all the vertices does not change the perimeter. This is the condition for a maximum or minimum, and from geometrical considerations it is seen to be a maximum.

By the last two propositions it is seen that the same portion of a polygon circumscribed to an arc of the first ellipse and inscribed to an arc of the second enjoys, at the same time, in respect to its perimeter the properties of maximum and minimum: the same as a *regular* polygon inscribed to an arc of a $\odot$ and circumscribed to an arc of a concentric $\odot$. Thus in the matter of maximum and minimum *perimeters*, concentric circles and confocal ellipses correspond.

Prop. VII. If *areas* instead of perimeters be considered, then as before, under similar conditions, the maximum polygon in an arc is the regular polygon, and the minimum area circumscribed is that of the regular polygon. Then since facts concerning areas are preserved in the projection we see that, since concentric $\odot$'s project into concentric and similarly placed ellipses, the maximum area is that of the regular inscribed polygon, and at the same time the minimum area is that of the regular circumscribed. Thus for perimeters, confocal conics correspond to concentric $\odot$'s, and for areas concentric and similarly placed ellipses correspond to concentric $\odot$'s.

Prop. VIII. When two arcs of an ellipse are similar if we circumscribe to them the two portions of polygons of m sides, of minimum perimeter, the difference of the two perimeters is always the same for different values of m and equal to the difference of the arcs.

For example take $m = 3$. Then if (Fig. 7) HB and DE be similar arcs, it can be easily shewn that AC , CB , EF , and FD are similar arcs.

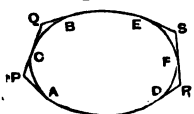


Fig. 7.

Then difference between EF and BC = difference between the sum of SE and SF and that of QB and QC , and so for FD and CA . $\therefore$ Difference between ED and HB = difference between $ES + SR + RD$ and $BQ + QP + PA$. $\therefore$ etc., and so for any number of similar subdivisions.

Prop. IX. All these properties have been stated for an *ellipse*. They can easily be shewn to hold for all confocal *conics*.

All of the propositions on the subdivision of arcs have an interesting meaning, when the arc of the ellipse so divided is the whole ellipse. Thus let the ellipse be divided into n parts, and the tangents at the points of division form a polygon then.

- (a) The vertices are on a confocal ellipse.
- (b) For a polygon of the same number of sides, corresponding to another point of departure for the division of the ellipse, the second curve is always the same.
- (c) The polygons have all the same perimeter.
- (d) The perimeter is a minimum with respect to all polygons of same number of sides circumscribed to the ellipse.
- (e) And a maximum with respect to all polygons of the same number of sides inscribed in the second ellipse.

Prop. X. Reciprocally, when a polygon of m sides is inscribed in an ellipse, and at the same time, circumscribed to a confocal ellipse, the polygon is, with respect to the first, of maximum perimeter, it is therefore such a polygon as we have been considering, and must be of minimum perimeter with respect to the inner ellipse: The m sides divide the inner ellipse into arcs whose differences are rectifiable, and an infinite number of polygons can be circumscribed to one and inscribed in the other and they all have the same perimeter.

The rest of the paper does not really introduce any new theorems, but rather discusses the application of those

already shewn, to some problems in geometrical optics. Interesting as they are, they are somewhat foreign to the purpose of this paper, and I shall not give them. The propositions adduced can hardly fail to be of interest, being as remarkable in their way as the famous theorem on which they rest. As already said in Chasles' paper the propositions are only stated. I shall be sorry if the notes of solutions I have added will make these elegant propositions appear tedious.

ALFRED T. DELURY.

Vancouver, Feb. 7th, 1891.

THE STRUCTURE OF MATTER.

By A. C. Chant, B.A.

Every year we hear of large numbers of people making trips at the expense of much money and a great deal of time, in order to look upon the far-famed dress of Nature, or to gaze at and study the wonderful productions, ancient and modern, of daring skilful architect or revered master-painter. And I do not think that this is to be deplored ; but, nevertheless, there is no reason why the most of us, who cannot afford such a great outlay, should sit aside and bemoan the fact ; for, if ever there was a true saying, it is the statement that all about us, beneath our feet, above our head, on the right hand, on the left—yes, everywhere—are to be found subjects which are as well worth our careful attention as in the loveliest combination of water, hill and dale that the earth can show, or, as is the most cleverly carved sculpture that the hand of man has produced.

I refer to the fairy-land of science, spread all about us. We can delve in its fields to ascertain how the rocks lie, or we can study what life is in the biological domain, or perhaps we might be pleased to learn according to what laws the motions and various actions of the different parts take place. But one of the most interesting questions is what the many palaces themselves of this fairy-land really consist of, and it is that which I shall discuss briefly in this paper.

I shall give the two theories which have been propounded to explain what matter really is, and after giving reasons for accepting one of these theories I shall briefly sketch how the approximate discussions of the ultimate particles of matter have been obtained.

There are two theories as to what is the structure of physical substances ; 1st, the doctrine of continuity ; 2nd, the hypothesis of molecules, or the atomic theory. The first asserts that as we can divide a drop of water into two other drops, so each of these parts can be continuously divided and redivided ; and there is nothing in the nature of things to prevent this division times without number. The disciples of the atom theory, on the other hand, contend that there is a limit to this physical division, and that the body remaining when this limit is reached is the molecule. The molecule, then, is the smallest particle of a body which retains the physical qualities of the body. If the molecule is still further dissected its nature is entirely changed, and we reach its constituent atoms, which, we thus believe, are the smallest possible divisions of an elementary substance. For instance, a molecule of water can be broken up into two atoms of hydrogen and one of oxygen, the chemical name being H_2O .

Both theories are very old. That of atoms is to be found in the writings of Democritus, who lived from 460 to 361 B.C. ; while Anaxagoras (500-428 B.C.) explained and defended the doctrine of continuity. The arguments for both sides are given in the first two books of "De Rerum Natura," by Lucretius the greatest of classic didactic writers, who died in 51 B.C., aged 44 years.

The atomists declare that the atom is the smallest possible division of a body : that, as its name indicates, it cannot be cut in two. The reply to this was that the smallest conceivable part of a body has parts, and whatever has parts may be divided. But this is a quibble, rather than an argument, and the answer can be easily given.

The theory of atoms, I believe is the generally-accepted hypothesis of the present day, but it is not very long since that few chemists embraced it ; and, indeed, it is only the scientific advances of the last half-century which have placed it on a firm basis—advances due chiefly to Clausius.

Maxwell and Sir W. Thomson, who have developed the kinetic theory of gases, and have been enabled to satisfactorily explain the various properties and phenomena of material bodies on the supposition of molecules and atoms.

I have referred to the distinction between molecules and atoms. By continuous subdivision of a drop of water the ultimate molecules will be reached. Each molecule of water is a combination of three atoms, held together by the great force of chemical attraction or affinity. Now every body is supposed to be similarly constituted, that is, it is an aggregate of these units. In the case of gases, however, the molecules have perfect liberty, and can move freely in every direction. As a matter of fact they are all in rapid motion, and as a result of their continuous assaults against the containing vessel, pressure is produced. Even the velocities of the ultimate particles have been calculated, the values being given as follows:

Oxygen,	461 metres	(1,512 ft.)	per sec.
Nitrogen,	492 "	(1,614 ft.)	" "
Hydrogen,	1,844 "	(6,048 ft.)	" "

In a liquid the molecules are free to move anywhere in the liquid, but must stop at the bounding surface, unless, indeed, heat is applied and the increased velocity thus given to the particles enables them to leap from the liquid, thus giving us evaporation. In the solid the molecules are not at rest, but the motion is perhaps oscillatory about a central position.

An illustration will perhaps best explain the two theories. A cubic inch of water vapourises into almost a cubic foot of steam. Now in the atomic theory, we suppose that the cubic inch of water contains a certain number of molecules, and when in the condition of steam the molecules remain exactly the same in number, but become far more widely separated from each other.

In the other theory it is supposed that the space within the faces of the cubic inch of water is absolutely filled with

matter, and when the remarkable expansion takes place the space within the cubic foot is still filled with a continuous mass of the substance.

Now which theory are we to accept? The results of a few experiments will, perhaps, best indicate in which to place our confidence. One is as follows :—

Take a globe of volume (Fig. 8) one cubic foot, and let there be attached to it a pressure gauge, a thermometer, and a stop-cock by which any substance may be introduced. Let the air within the globe first be exhausted ; and suppose the temperature be kept at 100° C. Place a cubic inch of

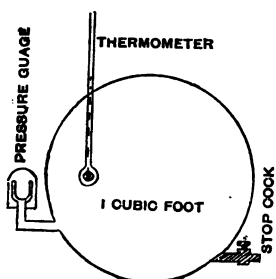


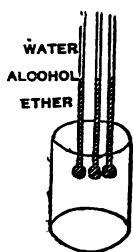
Fig. 8.

water in the globe; it is all vapourised. Add more water; no more steam is formed. Now read the pressure. Next introduce some alcohol; this evaporates, the vapour rising as though no steam at all was present, except in that it expands a little more slowly. Again read the gauge, which will give the pressure produced by the

two vapours. Still further add some ether, and again the vapour forms irrespective of the present occupants of the room. By a third examination of the gauge we obtain the aggregate pressure produced, and this pressure is found to be exactly the sum of the three pressures produced when the globe is filled by the three vapours in succession, in each case commencing with a perfect vacuum.

Now in this case there is no chemical combination ; it is only a mechanical mixture, and if the steam (or the water vapour) absolutely filled all the space within the globe some energy, at least, must be spent in forcing in the other vapours. Yet no more work is required than when the globe is empty. Hence we conclude that no vapour completely fills the space ; that between the molecules there are interstices into which the molecules of other substances may enter.

Another experiment can be performed by means of three thermometer tubes, with bulbs blown on the lower ends. Fill the first up to a certain height with water, the second with alcohol, and the third to the same height, with ether, and let them be placed within another glass vessel.



Into this vessel pour warm water, and observe the rate of expansion for each substance. In these cases it varies greatly. Then let the temperature be raised to the boiling point, thus volatilising the three liquids. It is now found that the expansions are much more rapid, but what is very peculiar, the rate of expansion in each is equal. Now, if the substances filled the

space why should there be discontinuity in this law of expansion? Also Avogadro's law states that, at the same temperature and pressure, in all gases there is the same number of molecules in equal volumes, and the latter part of the phenomenon is certainly confirmatory of this hypothesis.

A slab of glass certainly appears to us to be perfectly homogeneous; but did you ever think what happens when a ray of light is introduced into it? We all know how the prismatic colours are produced by passing through a prism a beam of light; and nearly half a century ago physicists were startled by a wild proposition of Cauchy's in which he stated that the distance from one molecule to its next neighbour must be comparable with the wave-length of light, or this familiar decomposition could not occur. The scientific advances since then have been quite corroborative of the substantial correctness of Cauchy's argument. A white beam we believe is a compound of coloured pencils of light, and these pencils on entering the medium have different indices of refraction, and hence they are separated on emerging. Now the velocity in a medium depends on the index of refraction, and therefore we see that the various coloured rays have also various velocities in the

medium. Now it is impossible to conceive how these different colours can have different velocities in the medium if it is perfectly homogeneous.

But from the undulatory theory of optics we can obtain some approximate measurements of the heterogeneity of these transparent solids. This, of course, assumes the correctness of the wave theory, which, perhaps, would not be admitted by all. There can be no doubt that the theory is a valuable one, and the power of predicting results by using it is certainly strongly in its favour. Lloyd's experiment I consider especially so. But for all that the numerous additional hypotheses which are resorted to to obtain double refraction, rotatory polarisation and other recondite parts of physical optics are certainly taxing on the imagination.

Perhaps a little calculation of what the ether is like will illustrate the difficulties in accepting the theory. Take the well-known formula:—

$$v = \sqrt{\frac{e}{d}}$$

where v = velocity of the disturbance in the air, and e is the elasticity, and d is its density.

Put v = velocity of sound per sec. = $\frac{1}{5}$ mile,

and v' = " " light " " = 192,000 miles.

$\therefore v' = v \times 1,000,000$ (approx.)

So that if $d = d'$,

$e' = e \times (1,000,000)^2$

where of course d' and e' refer to the substance through which light is transmitted, that is, the ether.

Now e is measured by the resistance which must be opposed to keep the atmosphere in equilibrium, or it is 15 pounds to the square inch.

Hence when $d = d'$

$e' = 15 \times (1,000,000)^2$ lbs.

$= 15 \times 10^{12}$ "

This is approximately the weight of a cubic mile of granite.

Now, suppose we take a cylinder, one square inch in cross-section, closed at the lower end, and in which a closely-fitting piston can be moved. If upon this piston we place a weight of 15 lbs. it will sink half way to the bottom. But, again, suppose we take an ether-tight cylinder—impossible, of course, but imagine it so—and place upon it this cubic mile of granite. The density of the ether will become only as great as that of the air at the earth's surface. This *may be* true, but we must accept it with some reserve—especially when we remember that this substance pervades every substance, and that the earth passes through it at the rate of a thousand miles an hour and yet experiences no sensible resistance.

But whether there be waves or not there is something which has been measured with very great accuracy, indeed with a precision as great as is reached in the calculation of astronomical dimensions, though at the other end of the scale. It will probably be difficult to conceive these as the magnitudes of anything other than *wave-lengths*. They are as follows :

LIGHT WAVES.

COLOUR.	NUMBER TO INCH.	OSCILLATIONS PER SECOND.
Red	39,000.....	477,000,000,000,000
Orange	42,000.....	506,000,000,000,000
Yellow	44,000	535,000,000,000,000
Green	47,000.....	577,000,000,000,000
Blue	51,000.....	622,000,000,000,000
Indigo	54,000.....	658,000,000,000,000
Violet	57,000.....	699,000,000,000,000

Now if the transparent medium were perfectly homogeneous there might be refraction, but there would not be dispersion ; but we must suppose the distance from molecule to molecule must be considerably less than a wave-length or the dispersion would be greater. It can scarcely be much less than $\frac{1}{10,000}$ as great, and remembering that a wave-length is about $\frac{1}{40,000}$ of an inch we calculate that from the centre of one molecule to the centre of its next

neighbour is about $\frac{1}{400,000,000}$ of an inch,—that is, we pass from molecule to nothing and from nothing back to molecule about 400,000,000 times in an inch. This is, of course, only a rough approximation.

For a long time there were associated with the idea of an atom incredible assumptions of infinite strength, absolute rigidity and infinite smallness; but the above reasoning would seem to show that these ultimate units are not infinitely small, but that they occupy definite spaces. This result, indeed, might almost be deduced from *a priori* considerations; for if the atoms are inconceivably small, why are not chemical actions indefinitely swift?

Another method of reaching a result is through experimenting upon the action of liquid films. Take a soap-bubble for instance. Using experimental data we can calculate how much work must be expended to expand the film to a certain size, that is, to reach a certain thinness. If we suppose this thinness to be about the four or five-hundred-millionth of an inch we calculate the work accordingly. Now take a square centimetre of this exceedingly attenuated film; we have the mass, and can hence calculate the work required to pull apart the molecules, that is, volatilise it. If the above thickness is that of a molecule, the two expressions for the work should correspond, and it is found that they do very satisfactorily. Hence we conclude that when a thickness of the four or five hundred-millionth of an inch is reached we have reached the diameter of a molecule.

Another very ingenious method is that founded on the electricity generated when two different metals are put in contact.

If a carefully-insulated plate of copper be connected to an electrometer, and then touched by a plate of zinc, the index on the electrometer at once moves, indicating the presence of electricity on the copper plate. It is also found that if a plate of zinc be connected with a plate of copper

by means of a small wire, the same phenomenon is produced, the zinc is positively electrified and the copper negatively charged. The difference of potential, according to Sylvanus Thompson is about .74 volts. Now as these plates are charged with opposite electricity they will attract each other, and a certain amount of work will be performed in bringing them close together. Sir Wm. Thomson, some thirty years ago, shewed that the attraction between parallel plates of two metals, held at a distance apart small in comparison with the diameters of the plate and kept connected as above indicated, would amount to two ten-thousand-millionths of a gram weight per area of the opposed surfaces equal to the square of the distance between the plates. It will be noticed that the thickness of the plates is of no consequence.

Now let a plate of zinc and another of copper, each one centimetre square, and $\frac{1}{100,000}$ centimetres thick be placed with a corner of each touching a metal globe whose diameter is $\frac{1}{100,000}$ centimetres. Let the plates, thus kept in connection with each other be at first wide apart (See fig. 10.)

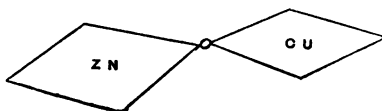


Fig 10.

and then let them be brought up under the influence of the attraction, until they are parallel to each other: that is to say we will have two plates, each one centimetre in area and $\frac{1}{100,000}$ centimetres thick, and separated by a space $\frac{1}{100,000}$ in width. (Fig. 11.)

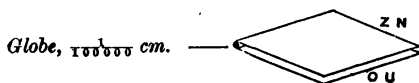


Fig. 11.

Now let us apply the above rule to find the force of attraction. The distance between the plates is $\frac{1}{100,000}$ cm.

and the square of this is $\frac{1}{10,000,000,000}$ sq. cm., that is in the surface of each plate is 10^{10} of these units. Hence the attraction is

$$\frac{2}{10,000,000,000} \times 10,000,000,000 \text{ grms. wt.} \\ = 2 \text{ grams weight.}$$

Again it can be proved by abstract dynamics and the theory of energy that the work done by the changing forces of attraction, during the motion by which we supposed this position to be reached, is equal to that of a constant force of two grams weight, acting through a space of $\frac{1}{100,000}$ cm., = $\frac{2}{100,000}$ grm.-cm.

Remember that this work is done on the metal, and takes the form of heat.

Now let a second plate of zinc be brought up by a similar process to the other side of the plate of copper; a second plate of copper to the other side of this second zinc plate, and so on until a pile is formed consisting of 50,001 plates of zinc and 50,000 plates of copper, separated by 100,000 spaces each being $\frac{1}{100,000}$ of an inch. It is found that the work done by the electrical forces of attraction in forming this pile is two gram-centimetres. The whole mass of metal weighs eight grams; hence we have this work, two grm.-cms. expended on eight grams of metal, that is $\frac{1}{4}$ grm.-cm. per gram. of metal.

Again, Joule's Mechanical Equivalent of Heat states that the amount of work required to heat one kilogram of water through 1°C. is a trifle less than 424 kilogram-metres, and transforming the units we find that the work necessary to be spent in raising one grm. of water 1°C. is 42,400 grm.-cms. The specific heats of copper and zinc are each nearly .095. Hence the work consumed in raising one grm. of the metal by 1°C. is $42,400 \times .095$, or, say, 4,030 grm.-cms. That is, 4,030 grm.-cms. per gram of weight must be expended on the zinc and copper in order to raise the temperature by 1°C.

But the work we expended above in building the pile was only $\frac{1}{4}$ grm.-cm. per gram; hence the temperature would be increased by only $\frac{1}{4} \times \frac{1}{4,030}^{\circ}\text{C}$, or $\frac{1}{16,120}^{\circ}\text{C}$.

But again let the thickness of each plate of metal and of each intervening space be $\frac{1}{100,000,000}$ instead of $\frac{1}{100,000}$ of a cm.; the work would be increased a million-fold, unless the thickness $\frac{1}{100,000,000}$ approaches the smallness of a particle. The heat generated would be, therefore, enough to raise the mass of metal by

$$\begin{aligned} 1,000,000 \times \frac{1}{16,120}^{\circ}\text{C} \\ = 62^{\circ}\text{C}. \end{aligned}$$

Yet still further suppose each plate to be made yet four times thinner, that is $\frac{1}{400,000,000}$ of a cm. The work would be increased sixteen-fold, so that we would have sufficient heat to warm the mass through 990°C . Now this is far higher than is required to fuse the metals to their brass alloy, much more than is produced by the metals when entering into a molecular combination. If, therefore, in reality anything like so great a heat as this were produced a mixture of zinc and copper powders, if melted at anyone spot, would run together generating more than enough heat to melt the whole throughout, just as a large quantity of gunpowder if ignited at any spot burns throughout without further application of heat. Hence we would infer that plates of zinc and copper of about $\frac{1}{300,000,000}$ cm. thickness if placed close together would form an approximation to a chemical compound, if indeed such thin plate can be made without splitting the molecules. So we see that this is an inferior limit; it is very likely that the coarse-grainedness is somewhat greater than this would indicate. Also it will be noticed that this agrees quite satisfactorily with the previous calculations.

Another method of approximation is from the behaviour of gases. From experiments and calculations of Clausius

and others on the velocity and the amount of compression which a gas will allow, it has been calculated that the diameter of a molecule is about $\frac{1}{500,000,000}$ of an inch. All of these methods agree satisfactorily, and we can, therefore conclude that they are in all probability fair approximations to the dimensions sought for. When combined Sir Wm. Thomson thus illustrates :

“Imagine a raindrop or a globe of glass as large as a pea (about $\frac{1}{8}$ in. diam.) to be magnified to the size of the earth, each constituent molecule being magnified in the same proportion ; the magnified structure would be coarser-grained than a heap of small shot, but probably less coarse than a heap of cricket balls.”

If we take as an average a small sized orange, we get from this approximation that as the orange is to the whole earth, so is the coarse-grained particle to the drop of water ; and if we stand at a distance at which an orange is just visible from a drop thus magnified we would be just able to detect its heterogeneity.

There are many other interesting points in the molecular theory ; but, begging pardon for my lengthy transgression, I must desist.

C. A. CHANT.

Ottawa, Feb. 21, 1891.

THE LAW OF HUMAN MORTALITY AND ITS PLACE IN SCIENCE.

By F. Sanderson, M.A.

What is meant by the phrase "Law of Mortality?" Is there a law of mortality in the same sense as there is a law of gravitation? If so, what is the mathematical expression, what its history and what its place among the sciences?

To give a superficial answer to these questions might be easy to some, to give a philosophical answer will require some study and consideration. The importance of the answers to the above questions will be recognized when it is said that they lie at the very foundation on which the great science of life contingencies is built, and hence also lie at the foundation of the principles of life assurance, the greatest financial institution of this age.

How often have we heard that thread-bare quotation from Dr. Southwood Smith, "Mortality is subject to a law, the operation of which is as regular as that of gravitation;" or that equally thread-bare one from Babbage, "Nothing is more proverbially uncertain than the duration of human life when the maxim is applied to an individual; but there are few things less subject to fluctuation than the average duration of life in a multitude of individuals."

Such quotations naturally lead one to the conclusion that the subject of mortality has arrived at the stage of an exact science, the same as the physical sciences--hence the importance of the present enquiry.

Let us first of all see what is meant by a law of nature, such as the law of gravitation. It is an hypothesis or supposition which accounts for certain phenomena noticed by us and which will in all probability, though not neces-

sarily, continue to account for the phenomena which we may notice in the future. It is simply that which gathers up and binds together our view of certain phenomena. It is the collocation or binding together in our mind certain facts of observation, analysis and experiment. That the sun revolved around the earth was for a long time believed to be a law of nature, until Copernicus exploded the hypothesis by which astronomers had bound together their view of observed phenomena, and showed this phenomena could be more rationally accounted for on the hypothesis that the earth and other planets revolved around the sun. Acting on this hypothesis of Copernicus, Tycho Brahe made numerous observations of the planets from which Kepler deduced his celebrated laws of the planetary orbits, Following up the labours of Copernicus, Tycho Brahe and Kepler, Newton put forth his great law of gravitation. (that every particle of matter in the universe attracts every other particle of matter, with a force which varies directly as the product of their masses and inversely as the square of their distance) an hypothesis which explained and will, in all probability, continue to explain certain observed phenomena of nature. If Newton's law of gravitation completely failed in the future to account for some newly discovered phenomenon, then it must fall to the ground and give place to another hypothesis, but this, though conceivable, is highly improbable.

Having then, got some idea of what is meant by a law of nature let us see what hypotheses have been put forward to account for the mortality among the human race, and if any of these hypotheses will rank as a law of nature in the same sense as the law of gravitation.

In the year 1725 De Moivre published in his "Treatise of Annuities on Lives" his famous hypothesis of equal decrements to human life, by which he supposed that the number of lives existing at any age is proportional to the number of years intercepted between the age given and the

extremity of old age—that is, that the column 1, (De Moivre's formula, $l_x = 86 - x$), forms a decreasing arithmetical progression.

This hypothesis did not account for the facts of observation, and in fact De Moivre does not appear to have claimed scientific accuracy for his hypothesis, from which, however, he believed tables of rates could be deduced with sufficient accuracy for practical purposes.

In the year 1825, just 100 years from the date of the hypothesis of De Moivre, Benjamin Gompertz contributed to the Royal Society of England a celebrated paper and one which has made his name famous in the history of life contingencies. As a result of that paper we have Gompertz' hypothesis as to the law of human mortality. He supposed that death might be the consequence of two generally co-existing causes:—the one, chance, without previous disposition to death or deterioration; the other a deterioration or increased inability to withstand destruction—a deterioration in which the average exhaustion of a man's power to avoid death was such that at the end of equal individual small intervals of time he lost equal portions of his remaining power to oppose destruction which he had at the commencement of these intervals. From this is deduced Gompertz' hypothesis, namely, that the force of mortality increases in geometrical progression. In mathematical language the law of Gompertz is expressed thus:—

$$\text{Force of Mortality} = Bc^x,$$

from which is deduced by means of the Integral Calculus the well known condition between the age and the number living, namely, $l_x = K (g^{c^x})$ (See appendix.)

This, then, is the law which for some years has been recognized as claiming attention by reason of its supposed scientific accuracy. The chief glory of Gompertz' theory is that it appears to be based on a physiological fact—namely, that the human constitution has the power of resisting the effects of disease, which power increases from

birth up to a certain age and diminishes from that time upwards.

This power of resistance Gompertz called the "Power to oppose destruction." The force of resistance and the force of deterioration of Gompertz seem to bear a remarkable analogy to (if indeed they be not particular examples of) the working of the universal law of evolution and the universal law of dissolution of Herbert Spencer's philosophy. According to Spencer the forces of evolution are forever struggling with the forces of dissolution and the state of equilibrium between them is invariably sooner or later reached. Societies, animal and vegetable life as well as individuals, according to Spencer, reach their period of "equilibration," after which the force of dissolution increases until "disintegration" takes place.*

If the above theory of Gompertz—that the human power to oppose destruction becomes (after the first few years from birth) ever less and less efficient—is a physiological fact, it must be stated his law of mortality is simply the assumption that the power to oppose destruction loses equal proportions in equal times, an hypothesis, however, which although not capable of a vigorous demonstration, yet appears highly probable both on *a priori* grounds and also as a result of observation; but although corroborated in a most remarkable way for a great part of life by the experience of life insurance companies it does not appear to apply to the younger ages of the life table and therefore fails to claim for itself that universality to which the law of gravitation is entitled. As just stated Gompertz' formula does not hold for young ages and a change of constants is in addition necessary between the ages fifty-five and sixty. With a view of giving greater exactitude to this formula, W. M. Makeham in 1860 introduced a modification which has the effect of making the formula apply with wonderful closeness to many mortality tables from about the age of twenty to the utmost limits of life."†

* "First principles."

† Assurance Magazine, 1860.

For Gompertz' formula Makeham writes,

$$\text{Force of Mortality} = A + Bc^x,$$

from which, by means of the integral calculus, we deduce the well-known formula, $l_x = lcs^x (g^{cx})$. (See Appendix II.)

Hence we see that, theoretically speaking, the law of mortality is a mathematical relation between the numbers living at different ages, so that having given a large number of persons alive at any age, it can be deduced by the law what number will survive any given number of years.

It has been said that the attempt to discover on philosophical grounds a law of nature applying to mortality is like searching for the philosopher's stone, which has never been found. As observed by George King, the accomplished Actuary of the Atlas Assurance Company of England, "some very plausible theories have been suggested for a law of mortality and very useful formulas deduced therefrom, but although in some cases (notably in the case of the Institute of Actuaries Mortality Tables) these formulas have been found to represent the table throughout its greater portion with abundantly sufficient accuracy for practical purposes, yet so far as it has not been possible to represent with absolute exactitude by a formula, any table derived from actual observations."

Before discussing further the question as to the possibility of the existence of a natural exact law of human mortality, it seems necessary to try and discover to what department of science our subject belongs and to what stage of development it has advanced, and then to enquire if there is any ground for believing that the future will reveal such further developments as to justify the assertion that there is a natural law of mortality which can be represented by a mathematical formula.

The subject of mortality is really a branch of the newly developing science of Statistics, which has made such rapid advances in recent years through the labours of the Royal Statistical Society of London, Eng.

The subject of Statistics is again a subdivision of Economics which latter is in turn included in the great and still undeveloped science of Sociology, whose foundation was first laid by Auguste Comte in his *Positive Philosophy*, published in 1839 and, in more recent years, extended by the labours of Schäffle in Germany, Spencer in England, and Ward in America.

It thus appears that the possibility of the existence of a law of human mortality is involved in the larger question as the existence of natural laws in the science of Economics.

Now any one acquainted with the revolution that has been wrought in the theory of economics within the last half century, will be aware that endless discussions and controversies have been waged as to the proper method and logical character of economics, the old school of economists, including Smith, Malthus, Ricardo, Mill, and perhaps Cairnes, holding that economics was a deductive science, while the new or historical school, including Comte in France, Roscher, Hildebrand, Knies, Schäffle, Wagner in Germany, Cossa in Italy, and Jevons in England, and Carey in America, have broken entirely away from what the late W. Stanley Jevons calls the "fools paradise," in which the old school lived, and have endeavoured to demonstrate that economics is an inductive science in a formative stage.

If economics is a deductive science deduced from assumed axioms, then its subdivision, statistics, is also; and it would therefore seem to follow that the existence of a mathematical law of human mortality was a chimera, and a vain hunting for the philosopher's stone; but, if as now seems to be established, economics is an inductive science, in which, as in the organic sciences, any change of one element involves or determines a change in the other, and in which social phenomena follow regular tendencies, which can be accurately determined beforehand,

then the inductive method would also apply to statistics, and to its subdivision mortality.

A student of physical science is so accustomed to dealing with exact, rigorous laws of nature that at first sight it might seem absurd to attempt to apply the term natural law to the rate at which persons die, and like social phenomena, but it must be remembered that the physical sciences are largely hypothetical and are obliged to limit themselves to stating laws which are hypothetical and inconsistent with actual phenomena, though this inconsistency is no proof of the inaccuracy of scientific investigations. For example according to the principles of Dynamics the curve traced out by a projectile is a parabola whereas this is only hypothetically true.

In order to make physical science a possibility it is necessary to assume for example, perfectly rigid bodies, which nothing will bend or strain, or perfectly smooth surfaces, etc., which of course we do not ever expect to find in actual nature. So in social phenomena we have to try and eliminate if possible certain disturbing, complicated and incommensurable facts thus making the science a hypothetical one.

But there is one consideration which does demonstrate the inferiority of social phenomena to the physical sciences and that is that in physics it is always possible to eliminate disturbing elements, whereas in social phenomena it is not, for the latter are partly physical, partly physiological and partly psychological, and it is only when dealing with large quantities and a very large number of tabulated results that the psychological element is approximately removed.

As the distinguished Belgian theorist Quetelet says, "All observation leads to confirm the truth of the proposition that collectively considered, that which concerns the human race is of the order of physical facts. The greater the number of individuals the more completely does the

will of the individual (the psychological element), disappear and allow the series of general facts which depend upon the causes by which society exists and is preserved, *to predominate.*"

A few examples will illustrate this fact:—

Nothing at first sight ought to be less regular than the march of crime. Nothing, above all, ought more completely to defy human foresight than the number of murders which would arise without provocation and in encounters altogether accidental. Nevertheless experience proves that murders are annually nearly the same in number and that certain classes of crime are more prevalent at one age than another. Thus is presented to us the unfortunate sight of the reappearance of the same crimes, in the same order, and assuming in minute details the same form.

Again, one would naturally say a man was a romancer who would affirm that, viewed on a sufficiently large scale, such an act as suicide was subject to definite laws and is capable of almost exact calculation, and that too for each sex. But such is the fact. In the twenty-six years (1861-86), the total number of female suicides in Denmark was 2,803; the number that was calculated beforehand was 2,809, a difference of only six.

The various modes of death are calculable. For the same period the actual number of suicides by hanging was 8,108 for men, 1,568 for women, whereas the calculated number was 8,107 for men, 1,569, a difference of one only in each.

The very number in a given month may be calculated with scarcely less exactness. Thus during the same period of October the actual numbers were 1,953 for men, 573 for women, while the calculated numbers were 1954 and 574, a difference of only one in each. It must be pointed out, however, that these anticipatory calculations only deal with certain outward tokens of the social life. We may reckon the number of committed murders, but not of attempts to murder; one may reckon the number of mar-

riages that will actually take place, but we cannot reckon the number of cases in which people would have been glad to marry but for different causes have deemed it inexpedient to do so.

And so one might go on and give similar instances of what appear to be natural laws of statistics, of education, pauperism, of labour, of health, of trade, of agriculture, or manufactures, in all of which, however, there is so much subtleness, so many causes at work to produce the result, that it is difficult to come to definite and accurate results.

But time and thought will develop general principles and as the same principles of observation and experiment lie at the foundation of all inductive sciences it seems probable the department of statistics, which is still in its infancy, will at length be so developed and perfected by the application of the calculus of probabilities as to be entitled to a place among the physical sciences, and it has been predicted that "in a future century mathematical statistics will solve problems of the bare enumeration of which we have no idea."

Of all departments of statistics the subject of mortality has made the most progress, and although as previously indicated the existence of a natural law of human mortality has not as yet been finally and definitely established, it is possible that by the application of the theory of errors of observation which has been of such great service in the intricate and delicate calculations of Astronomy, occasional deviations and disturbances resulting from artificial causes will be eliminated.

As yet very little application has been made by actuaries of the subject of Least Squares to the mortality investigation. But as one of England's most distinguished actuaries has lately said, "Some more fortunate investigator, profiting by the errors of his predecessors, will no doubt be found who will show us conclusively what is the true method of procedure in this case, and until he ap-

pears we must be content to suspend our judgment upon questions at issue."

But in the meantime it has been proved by extensive observations on insured lives taken at different times and in different parts of the world that if Gompertz' law is not the exact expression of a law of nature, it at least approximates so close to actual experience on human mortality that it has been found sufficiently accurate to deduce tables of premiums therefrom on which to provide for the twelve billions of dollars of life insurance at risk in the world to-day.

APPENDIX I.

To Deduce Gompertz's Formula.

Let l_x = number of persons living at age x , according to a monthly mortality table.

Then l_{x+t} = number living at the advanced age $x + t$.

$l_x - l_{x+t}$ = number of deaths during t years, or t intervals.

$\frac{l_x - l_{x+t}}{t}$ = number of deaths in one year, or in one interval.

$\frac{l_x - l_{x+t}}{t}$ = rate of mortality in interval t , since t is a continuous function and an increment of x .

$Lt \left(\frac{l_x - l_{x+t}}{t} \right)_{t=0} =$ instantaneous rate of mortality or force of mortality.

$$\begin{aligned} \therefore \mu_x &= Lt \left(\frac{l_x - l_{x+t}}{t} \right)_{t=0} \\ &= \frac{-d l_x}{l_x dx} = \frac{-d \log l_x}{dx} \end{aligned}$$

Hence, according to Gompertz's assumption,

$\mu_x =$ force of mortality.

$= Bc^x$, where B and c are constants to be determined from the mortality table in question, and x the age.

$$= \frac{-d \log l_x}{dx}$$

$$\therefore d \log l_x = -Bc^x dx$$

$$\therefore \log l_x = -\int Bc^x dx$$

$$= \frac{-Bc^x}{\log_e c} + \text{constant}$$

$$= \log_e g.c^x + \log_e k$$

$$\therefore l_x = k.g^{c^x}.$$

This is Gompertz's celebrated formula.

APPENDIX II.

To Deduce Makeham's Formula.

In place of Gompertz's assumption,

$$\mu_x = Bc^x,$$

Makeham assumes

$$\mu_x = A + Bc^x,$$

where A, B, c , are constants to be determined from the Mortality table in question, and x the age.

$$\begin{aligned} \text{Hence } \mu_x &= Lt. \left[\frac{l_x - l_{x+t}}{t} \right]_{t=0} = \frac{-d l_x}{l_x dx} \\ &= \frac{-d \log l_x}{dx} = A + Bc^x. \end{aligned}$$

$$\therefore d \log l_x = -(A + Bc^x) dx$$

$$\log l_x = -\int A dx - \int Bc^x dx$$

$$\therefore l_x = -Ax - \frac{Bc^x}{\log_e c} + \text{constant}$$

$$= x \log_e s + \log_e g.c^x + \log_e k$$

$$\therefore l_x = k.s^x g.c^x$$

This is Makeham's formula.

To determine the constants in Makeham's formula :

$$\begin{aligned}
 (a) \log l_x &= \log k + x \log s + c^x \log g. \\
 \log l_{x+t} &= \log k + (x+t) \log s + c^{x+t} \log g. \\
 \log l_{x+2t} &= \log k + (x+2t) \log s + c^{x+2t} \log g. \\
 \log l_{x+3t} &= \log k + (x+3t) \log s + c^{x+3t} \log g.
 \end{aligned}$$

Differencing both sides of the above equations twice, according to the principles of finite differences :

$$\begin{aligned}
 (\beta) \Delta \log l_x &= t \log s + c^x (c^t - 1) \log g. \\
 \Delta \log l_{x+t} &= t \log s + c^{x+t} (c^t - 1) \log g. \\
 \Delta \log l_{x+2t} &= t \log s + c^{x+2t} (c^t - 1) \log g.
 \end{aligned}$$

$$(\gamma) (1) \Delta^2 \log l_x = c^x (c^t - 1)^2 \log g.$$

$$(2) \Delta^2 \log l_{x+t} = c^{x+t} (c^t - 1)^2 \log g.$$

$$\therefore \frac{\Delta^2 \log l_{x+t}}{\Delta^2 \log l_x} = \frac{c^{x+t} (c^t - 1)^2 \log g}{c^x (c^t - 1)^2 \log g} = c^t$$

$$\therefore \log (\Delta^2 \log l_{x+t}) - \log (\Delta^2 \log l_x) = t \log c.$$

Hence we find $\log c$,

and $\therefore$ " " c .

Substituting for c in (γ) we get $\log g$ and $\therefore g$.

Substituting for c and g thus found in (β) , we get $\log s$, and $\therefore s$.

Substituting for c, g, s in (a) we find $\log k$, and $\therefore k$.

Hence the constants k, c, s, g are all determined.

NEWTON'S LAWS OF MOTION.

By R. Henderson.

In reading dynamics there were some difficulties which met me at the very threshold, or rather in the foundation of the science. For I read for some time and thought that I thoroughly understood the basis of its reasoning. But some remarks of Mr. Witton, late Fellow in Physics, led me to examine more minutely the statements of Newton's Laws of Motion, and I found there terms used which had not been defined, and of which I could find no satisfactory definition. Accordingly I was for some time in doubt as to the accuracy of the results, until a more thorough investigation of the meaning of the laws and of the evidence on which they are based, together with an enquiry into the form taken when these undefined terms were eliminated, shewed that this apparent lack of accuracy did not affect the accuracy of the results; and that the laws along with some assumptions, that were legitimate and consistent with themselves in view of the laws, formed a substantial basis for the superstructure which had been built upon them. I, therefore, thought it worth while to set down in order the results of my enquiry, that I might have them by me for future reference.

The first law I found stated as follows: "Every body continues in its state of rest, or of uniform motion in a straight line, except in so far as it may be compelled by impressed forces to change that state." The difficulty in this lay in uniform motion. What is uniform motion? I found that a body was said to be moving uniformly when it passed over equal spaces in equal intervals of time. But how will we measure equal intervals of time? What constitutes equality in two intervals of time?

There are three quantities which we must be able to measure before we can perform any experiments in physics. There are mass, space, and time. On the correctness of our measurements of these depend the accuracy of our computation of other physical quantities for every other quantity of which physics can take cognizance, is expressible in terms of these three quantities. In order to measure these quantities, certain assumptions must be made, and we must be careful that our assumptions are in accordance with the laws of nature.

But here is a difficulty. How can we know whether our assumptions are consistent with the laws of nature without knowing those laws; and how can we find out these laws without being able to measure space, mass, and time? Let us enquire. In the first place, what assumption do we make in order to measure equal distances? To answer this we must enquire what are the means we use to measure distances. We use rules and scales, and we see that the assumption made, is that the distance between two points of a rigid body is unaltered by any change of place, time or direction. Assuming this, we apply our scale to any distance which we wish to take as a unit and mark two points on it at the required distance apart, and then by trial or geometrical methods, we divide it into as many parts as required. We then find the distance between two other points by applying the scale to them and finding the number of parts on the scale which corresponds to the distance between the points. The basis of the legitimacy of this assumption, is partly experimental, in that the eye perceives no change in size caused by transference and partly metaphysical, in that we see no reason why a change of place, time, or direction, should change their distance.

With regard to the measurement of mass the course is not so simple. If we examine the means used to measure mass we see that we assume properties of a balance the

proof of which depends on the laws of motion. But let us enquire in what manner certain units were established. The unit which is useful for this purpose is the French unit of mass, the gramme. We find that it is established to be the quantity of matter in a cubic centimetre of pure water at 4° Centigrade. Thus the measure of mass of water is made to depend on the measurement of volume which is readily seen to depend directly on the measurement of distance. The assumption here made is that the same mass of water occupies the same volume at the same temperature unaltered by time or place. This assumption is readily seen to be of the same nature as the last and to extend to all substances, so in any particular form of a substance the mass is proportional to the volume, other things being equal. Leaving at this point the measurement of mass, having arrived at a point where we can obtain two bodies of equal mass or of any given ratio of mass, examine now the measurement of time. The modern practical determination of time ultimately depends on the equal timed oscillation of a compound pendulum. There are now two courses open to us, either we may reject this as a theoretical method, using it only as a practical method based on theory, and seek some other assumption, or we may enquire whether on *a priori* grounds we may consider it a legitimate assumption. We take the latter first. We find that underlying our two former assumptions there is the all-pervading principle that the same cause always produces the same effect, or rather that if in two cases the chain of causes be the same in every respect, the effect will be the same. Extending this principle we find that it may be considered as a sort of basis of Sturm's definition of equal times, which is as follows: "Two intervals of time are equal, when two bodies, the same in every respect, placed in the same circumstances, traverse equal spaces in the two intervals of time, whatever be the law of their common motion." From this defi-

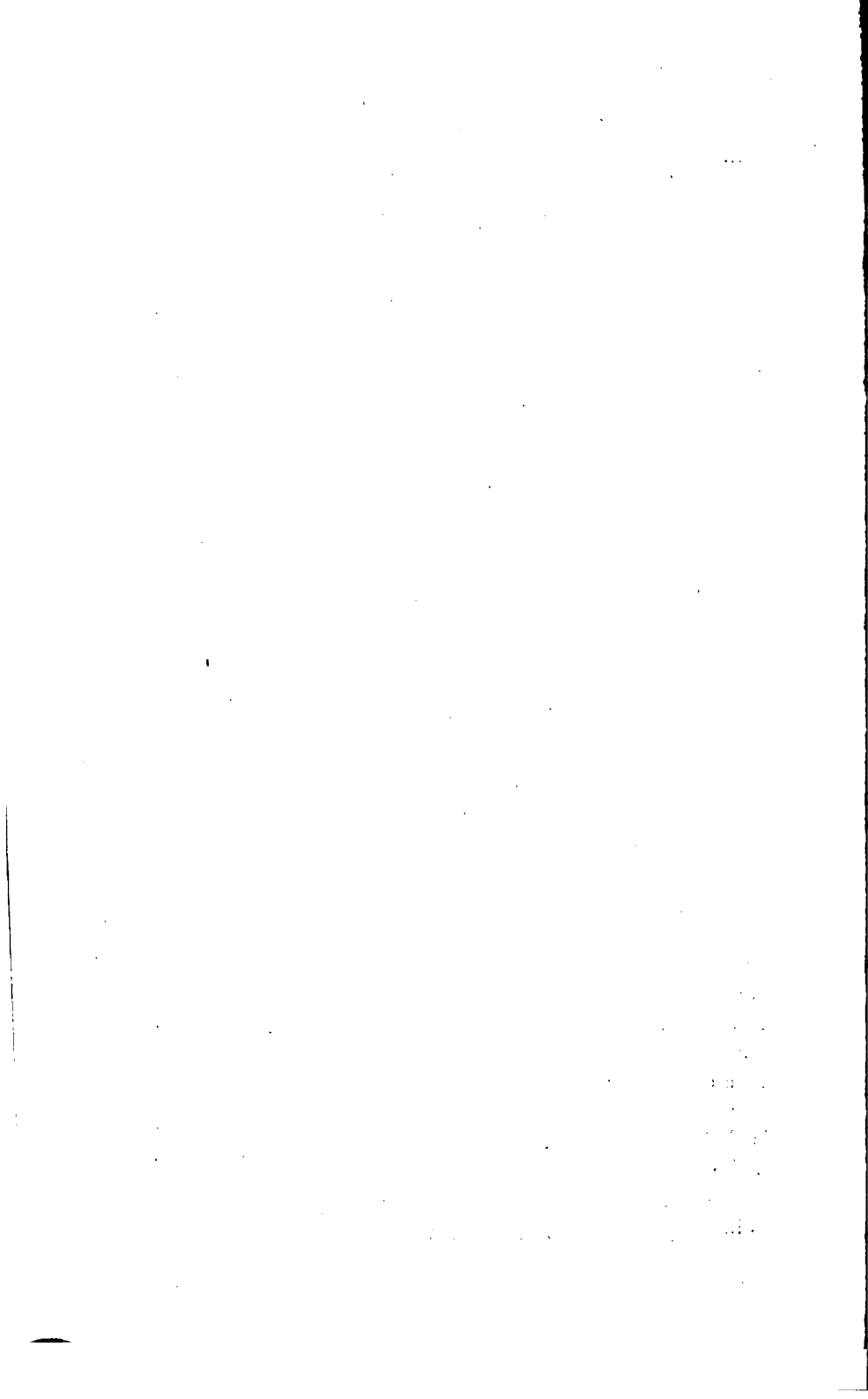
nition we see at once that the times of oscillation of a pendulum are equal, for when the pendulum has made one complete oscillation and returned to the place from which it started it is placed in the same circumstances as when it started the first oscillation, and consequently the time of the second oscillation is equal to that of the first, and so we may go on.

If we do not accept this let us enquire whether or no the measurement of time can be eliminated from the statement of the first law of motion. Let us state it in this form as a fact based on experience without any instruments for measuring the intervals of time, as follows:—
 “If two bodies are moving under no forces, they continue moving in straight lines, and the ratio of the spaces passed over in the same interval of time is constant no matter when that interval begins or ends.” Then let us define equal intervals of time as those in which a body under the action of no forces passes over equal spaces. This definition or assumption is consistent with the law, for no matter what bodies we take, their motions will agree with the definition.

Leaving here the first law let us examine the statement of the second and see if we can resolve it into simpler statements. I find it stated as follows: “*Change of Motion* is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.” I have thought if it were put in the more elementary form: “The same force always produces the same change of motion in the same direction in the same body, and this change is additive to that produced by other forces,” or taking the first part as a definition of forces equal in magnitude and direction consistent with the principle of cause and effect formerly laid down if we state the experimental law as follows: “The changes of motion produced by different forces are additive;” if, I say, we state it in this form we may deduce the law from it by mathematical methods. Change of motion is measured by the product of the mass into the change of velocity, and we may prove that n times the force

will produce n times the change of velocity in the same mass and the same change of velocity in n times the mass.

This brings us again to the measurement of mass, and we seek means of comparing the masses of two bodies of different substances. We know that equal forces produce equal accelerations on equal masses of the same substance, and we extend this and assume that those masses of different substances are equal on which equal forces produce the same accelerations. This points to a practical method of determining mass, for it may be proved mathematically from the laws of motion that the forces exerted on the two ends of an even balance must be equal to one another, but these must be equal to the forces of gravitation on the two bodies suspended, which produce equal accelerations on the bodies, being each the acceleration of falling bodies, therefore the masses must be equal. Thus we see that what we really do in weighing is to determine the ratio between the attraction of gravitation on the body and that on the given standard mass. But this latter is known to vary for different parts of the earth, so that we cannot determine absolutely by this means the force of the attraction of the earth on the body. But since the ratio of the forces is the same as the ratio of the masses, this latter ratio is determined and therefore, since the mass of the standard is constant, the mass of the body is determined, and since this is constant we may rest assured that a ton of coal will not change its weight by one grain though it be carried from the equator to the north pole. By weight here I mean the weight as found by an ordinary pair of scales. Of course if we attempt to measure weight by the bending of a spring we will find the spring more bent at the north pole than at the equator, and less bent on the top of a mountain than at the level of the sea. The reason of this is that the spring measures directly the force of gravitation, while the ordinary scales measure merely the ratio between two forces. So that while the former varies from point to point the latter remains constant.



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MUSICAL SCALES ;

THEIR ORIGIN, FORMATION, AND THE PHYSICAL RELATION
WHICH THEY BEAR TO MUSIC.

BY W. J. LOUDON, B.A.

WHEN one turns over the pages of ancient history and endeavours to unravel the mystery attending the origin of our musical scales of to-day, one meets at first with many difficulties: such varied accounts and such improbable stories are told, even by the truth-loving and philosophical Greeks, that the task of sifting out the truth becomes rather tedious; and yet, far back as we may go, we find crude notions of music and of a musical scale. Indeed, we may even suppose, that in the days of pre-historic man, our intermediate ancestors took their first music lessons from Nature: from the winds as they whistled softly through the forest pines: from the innumerable brooks and streams and rivers with their ever-changing melodies: or from the birds who doubtless sang just as sweetly as they do to-day. Nature doubtless was our first music teacher. We know that birds sing perfect intervals of our diatonic scale: our Canadian robin, the whip-poor-will, and canary, and many others sing in thirds and fourths; and it is not irrational to suppose that man made his first attempts in music by imitating the songs of birds.

Later on, he must have taken naturally to whistling and rude forms of wind instruments, and finally to instruments with stretched strings. But we learn so slowly and assimilate so feebly, even in our advanced civilization of the nineteenth century, that man's progress in the musical art three or four thousand years ago, must have been exceed-

ingly slow. In fact, we find nothing definite about a musical scale until the time of Pythagoras. That the Chinese, the Ancient Britons, the Egyptians, and the Hindoos were acquainted with the rudiments at least, of a five-toned scale, we are well aware, but we meet with nothing resembling our modern diatonic scale of eight notes, until the middle of the sixth century B. C.

Investigators who have probed into the past, tell us of an ancient Chinese scale of five notes corresponding to our *F, G, A, C, D*, of similar arrangement to the old Scotch scale; of the Arabians, with their scale of unequal temperament, and their complicated system of half-tones and third-tones, and a still more complicated music; of the Hindoos, who went still further and used quarter tones, involving further complications; but the scales attributed to the ante-Grecian era, sound so barbarous to our ears that one is loath to believe any statement which calls them by the name of *musical*.

The character of all music, prior to Grecian, must have been extremely simple, and the airs, we may infer from history, resembled somewhat the weird Hungarian airs one often hears now played in the key of A minor. From all we can learn it was never customary amongst any ancient people to express their *feelings* in music: the musical art was looked upon, even by the Greeks, as of secondary importance to poetry and declamation: the Chinese, even at the present day, never express sentiment by music, a good reason why their music sounds so harsh to western ears.

But when we come to the history of Greek music we find amidst the chaos of Greek scales several unmistakable landmarks. Whatever they may have known before 560 B. C. it is certain that Pythagoras, who flourished about that time, gathered together the fragments of different systems and reduced them all in some way to a common scale.

He was a man thoroughly imbued with musical sentiment, and is said to have regarded music as celestial and divine, inasmuch as he caused his disciples to be awakened every morning and lulled to sleep each night by the sweet strains of an *Æolian* harp. Various stories, most of them untrue, are told about his discovery of the Pythagorean scale, which is generally looked upon as the origin of our major diatonic scale. He seems, at all events, to have discovered the idea of ratios, and by using a monochord, (a single stretched string with a moveable bridge) to have obtained the octave, fifth, and its inversion, the fourth.

Whether he developed the Pythagorean scale from experiments on the monochord or by a combination of tetrachords, is not certain: he left us, however, a musical legacy in the scale which bears his name, and which, with but slight alterations, has come down to us to-day, and forms the basis of our major diatonic scale.

It is but fair to mathematicians to add that Euclid and Ptolemy wrote works upon musical scales. The former reduced the speculations of Pythagoras to a definite idea, and had a good insight into the formation and difficulties of Greek scales; and, moreover, evidently had some knowledge of temperament; but in his writings he is mathematically close and dry, and leaves no rules for the guidance of others.

Ptolemy was a musical butcher. He delighted in the manufacture of abnormal scales, twisting and dissecting the intervals in all possible ways, and framed three scales of his own. He is, however, to be remembered on account of the excellence of his writings and his thorough knowledge of the Greek scales.

From the Pythagorean scale six others were developed: calling the notes of this scale, *c, d, e, f, g, a, b, c*, let us take the scale, *d, e, f, g, a, b, c, d*, ... and so on. We can evidently form seven scales by taking the notes in order; and this is the manner in which the Greeks arrived at their seven scales.

Reduced all to the same tonic C they would be written :

Lydian	C	D	E	F	G	A	B	C
Ionic	C	D	E	F	G	A	B $\flat$	C
Phrygian.	C	D	E $\flat$	F	G	A	B $\flat$	C
Eolic	C	D	E $\flat$	F	G	A $\flat$	B $\flat$	C
Doric	C	D $\flat$	E $\flat$	F	G	A $\flat$	B $\flat$	C
Mixolydian	C	D $\flat$	E $\flat$	F	G $\flat$	A $\flat$	B $\flat$	C
Syntonolydian	C	D	E	F $\sharp$	G	A	B	C

As to the character of Greek music it was unquestionably simple. The Greeks had no idea whatever of harmony even although possessors of instruments with many strings and for this reason, if for no other, it is almost inconceivable how Pythagoras could have discovered his scale from experiments on strings, and not have been led to perceive the elements of harmony.

Their choruses were usually made up of male singers who sang the air and of boys who sang in octaves above thus merely heightening the effect ; and as stated before they seemed to regard music as an auxiliary to poetry, tending to increase the effect of the words.

There are, of course, many staunch supporters of Grecian music who maintain that their solo-singing was exquisite ; but any impartial investigator must come to the conclusion that were it possible to drop in to an evening performance of one of the old Greek plays with choral accompaniment, as performed two thousand odd years ago, he would be very apt to place their choral music in the same class as the psalm-singing of a band of our North American Indians.

Finally, the Greeks had no notion of a tonic or key. To them all keys were the same and although it is said that a passage in Aristotle points to the idea of a key and key-note, yet the very construction of their scales disproves this statement and shows us that they must necessarily have had no insight whatever into our modern conception of music written in an individual key.

Whatever their merits or demerits as far as harmony goes, the Greek scales flourished in Italy under the Romans, modified but slightly by Etruscan ideas. The Romans were too martial, too utilitarian, a people to think much about music, and were quite satisfied with what had come from Greece.

And it was not until the sixteenth century that attempts were made to modify in any way the old Greek scales.

With the gradual development of the organ and stringed instruments came necessarily an investigation into the difficulties of musical scales: and although mathematicians came to the rescue and constructed out of the Pythagorean scale our major diatonic and minor scales, defining accurately the ratios which represent different intervals yet the difficulties of arranging mechanism for instruments to play in various keys, became so great that a climax was soon reached. On the organ we find various systems of tuning adopted to allow of fingering in different keys, notably the mean tone system, which was a sort of unequal temperament. These were, however, only paving the way for the modern "scale of equal temperament."

To understand exactly what was happening, let us consider the major and minor scales as now defined by physicists, and differing but little from the Pythagorean scale.

These two scales are obtained mathematically by commencing with the pitch of a note (a number representing a certain number of vibrations in a second) and multiplying that pitch by certain fractions:

$$\begin{array}{l} \frac{9}{8}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{15}{8}, 2 \text{ for the major.} \\ \frac{8}{9}, \frac{6}{5}, \frac{4}{3}, \frac{3}{2}, \frac{5}{4}, \frac{15}{8}, 2 \text{ for the minor.} \end{array}$$

Here we have a definite idea of musical scale, and as far as perfect instruments are concerned, commencing at any note, the following ones can be obtained, and wherever we commence we can always obtain the true intervals by strictly observing in same way, by the ear or otherwise,

the above fractions. But perfect instruments are rare—the two most familiar to us being the human voice and the violin. With them, all notes and gradations can be produced (within certain limits), and the ear can be educated to recognize the perfect intervals of the above scales as represented by the ratios. Such instruments can bring forth music which is perfect from a scientific as well as an artistic point of view. To those who have heard a violinist play a simple melody in intervals of a perfect scale, or who have listened to a voice singing unaccompanied by an equal temperament piano, it is not necessary to say much about the beauties of the scale and intervals as above defined.

Unfortunately for us, we are not content with perfection: we cannot all lift our voice in song, nor can we spend half our lives in learning to play perfect intervals on the violin. We need something simple and easy, even though it be imperfect; and during the last century attention has been entirely directed to the making of instruments mechanically easy to play, and capable at the same time of being played in various keys. From all this has resulted the production of stringed instruments with frets, the piano and organ, all arranged for what is known as the scale of equal temperament, a scale which is almost entirely used at the present day.

Let us, in order to appreciate the meaning of this scale, examine the necessities of a stringed instrument such as the piano. Let us commence at middle *C* on the piano and form the true diatonic scale of physicists by tuning eight strings according to the ratios expressing pitch. This, of course, can easily be accomplished, and we could have a series of octaves (all white notes for example) in perfect order. But suppose now we wish to play in the key of *D* perfect intervals, it is evident we must introduce new notes which will not all correspond with those in the key of *C*; and, as we proceed from one key to another,

we find additional notes necessary for each change of key.

We find that in order to play perfect intervals in all the major keys, eighteen notes would be necessary, and if we introduce the minor keys, we find seven more needed. This takes no account of flats and sharps; but as modern music is written in flats and sharps, it can be seen that the number of notes to the octave will be so great that there would be no possibility of fingering them.

Many attempts were made to overcome this mechanical difficulty by reducing the number of notes on the keyboard; but the only method which allowed of playing in all the keys, was the one finally adopted. In the scale of equal temperament, the octave was defined to be twelve semitones, and as the pitch of the octave is twice that of the fundamental, a scale of thirteen notes in the octave was chosen with twelve intervals, the pitch of each being obtained from the next lower by multiplying by the twelfth root of two.

By comparison of the equal tempered with the true scale we can see slight variations, and yet they are so small that the ear, unless trained, barely notices them. At the present day, when the piano is so much in use, it is worth while remembering that the scale used is one of compromise, and that, although the ear can be trained to prefer the scale of equal temperament to all others, the process is one of imperfection. And what makes matters worse is the fact that not two pianos, even from the same factory, are ever tuned so as to give the pitch of each note the same in both pianos.

The process of tuning by fifths and fourths allows great latitude to the tuner, and tuning is something which depends, at present, on the individual taste, as well as on the accuracy of ear of the tuner. No plan has yet been devised for reproducing on the piano, by ear tuning, the intervals of the scale of even temperament, and, instead of having even the imperfect intervals of this scale, we must

allow our ears to submit to a second order of imperfection arising from the difficulties of tuning.

Various physicists, during the past few years, have laboured to accomplish something in the way of reformation; but the task has proved hopeless, and the piano, with its many imperfections, has become so firmly established in the musical world that it will, perhaps, be a thousand years before a reformation comes; and then, perchance, the musical critic of a more advanced age will look back with pity on the musical imperfections of the benighted people of the nineteenth century.

Something might be done in the matter of pitch, however: piano-tuners might at least adopt a common pitch from which to start, and if the different factories throughout Canada, for instance, would adopt a common pitch, we would, at least, have arrived at a first stage of happiness; but where, in addition to pianos being differently tuned through their intervals, we find sometimes one tuned a half or quarter tone above the other, the feeling is one of despair.

Turning to the case of fretted stringed instruments, which have lately come into prominence, we meet with more satisfaction. In the case of the guitar and mandolin, for example, which are strung with strings of different materials stretched over a finger-board with fret work, it is possible, at least, to obtain the perfect intervals of the scale of equal temperament. The open strings being of definite length, it is only necessary to apply the law that the pitch varies inversely as the length, obtain by calculation the necessary lengths to give the intervals of the scale of equal temperament, and insert the frets in the proper places.

As regards the utility of the equal tempered scale, it possesses many advantages, and certainly has developed musical taste during the last half century, and, although it is a compromise and brings a long train of evils with it,

yet we must admit, on investigation, that without it harmony would not have developed as it has, and all the modern composers of the German school, at least, with their intricate arrangements and muscular music, would have been like flowers "born to blush unseen" without the advent of the scale of even temperament.

Let us now examine a little more closely the major diatonic scale. Why does a succession of eight notes please the ear? Why are they connected to one another by mathematically defined ratios? Is there anything mysterious in the numbers or fractions which seem to define in a rigid constrained way that which lies at the basis of what has become the most elevating and a sentimental art in all the world? Is there anything sentimental or elevating in fractions? What connection exists between music and mathematics? Are mathematicians musical or musicians mathematically inclined?

These are questions which Pythagoras asked his disciples two thousand years ago; and they would probably never have been answered but for the investigations of physicists, upon whom it devolved to prove the relationship, and if possible bring about some union between the most widely different subjects in the whole universe, between music which appeals to the heart, and mathematics which appeals to the head.

Thanks to the investigations of Helmholtz, König, Ellis and others of less note; head and heart have been joined together, and the theory of music and harmony placed on a proper scientific basis.

Music has been wedded to mathematics and their offspring is harmony.

To explain how what might be otherwise regarded as an unholy and unnatural alliance took place, let us consider what is known as the Harmonic scale.

It is a scale defined in the following way. Any pitch may be chosen, as in the diatonic scale, on which to com-

mence; the harmonic scale is then obtained by multiplying this pitch by the numbers 1, 2, 3, 4, 5, etc., continued as far as we please. We thus obtain a series of notes which, taken either separately or all together, or combined in any way we please, usually produce a sensation of rest in the ear. For this reason the Harmonic scale is sometimes spoken of as the natural scale.

These notes of the harmonic or natural scale, are also spoken of as harmonics, and it is customary to call the starting note the first harmonic and so on.

To the French mathematician Fourier, we are indebted deeply for all we know of the laws of these harmonics when they appear as auxiliaries to the fundamental note of a stretched string; and we may even go so far as to say that by his beautiful and elegant theorem, he has done more for the advancement of music and the development of true harmony than even Pythagoras. In his theorem, which is known now to all mathematicians and physicists by the name of "Fourier's Theorem," he tells us that a stretched string vibrating freely between two points, is undergoing a process and passing through a cycle of changes, which can be recognized by the ear. That instead of vibrating simply and producing one note, as is generally supposed, the string gives out an infinite number of musical sounds: and these sounds, wonderful to relate, are nothing but the notes of a true harmonic scale defined by the natural series of integers, (1, 2, 3, 4 ...).

Any string then, if made of proper material, when plucked, will emit at least half a dozen different tones, all easily perceived by the ear. And since the pitch of a note varies inversely as the length of a stretched string, it is evident that the lengths of string required to produce these harmonics, will simply be $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, ... of the open string. Moreover, it can easily be shown by experiment that a string can be made to produce harmonics by division into two, three, four, or more parts, with points of rest or

nodes, as they are called ; and we are immediately led to the conclusion that according to the manner of plucking and the material of the string, some harmonics may be made more prominent than others.

Considering then a string vibrating freely and naturally, we find that the ear is quite satisfied with the effect produced on it ; the sensation is not disagreeable and certainly not discordant. We conclude then with Fourier, that the ear must be satisfied with the notes of the natural scale, and, therefore, for example, commencing on C, the ear will be content to hear the octave above, it will be satisfied to hear the next G, a fifth above, then C, then E, and so on. Commencing on another note, the same arguments will hold, and since the ear approves of all these intervals in the free string, there is no reason why it should not be content with the effect produced by different strings tuned to present a portion or portions of the natural scale. From all this follows at once the principles of Harmony in Music, and all modern theories of harmony commence by approving of the intervals on the harmonic scale. We will not enter the sacred precincts of musical harmony : it is enough to see that the foundation of a musical education should commence with a slight knowledge of the laws of vibrating strings. And if musicians despise mathematics, they should, at least, acknowledge the debt they owe to Fourier. On the other hand it is certain that mathematicians do not despise music, and when music again requires the aid of science the latter will be only too glad to hold out a helping hand. With regard to harmonics in strings, it is well to remember that the quality of tone, what the French call *timbre*, depends entirely on the number and intensity of the harmonics or overtones. Silk strings and those made of gut give a good quality of tone from the fact of the presence of the first three or four harmonics and the absence of most of the rest. Steel strings yield a good quality of tone when pro-

perly plucked, but are apt, when improperly struck, to yield disagreeable high ringing overtones.

And now to revert to the intervals of the diatonic scale. Why do we like the succession of eight notes? The answer to this question is that it is a result probably of education of the ear, and doubtless the ear could be cultivated to admire the sequence of the first eight notes of the harmonic scale. If, however, we now ask the question why are the pleasing intervals defined by ratios? we can answer at once, because all these intervals are contained in the natural or harmonic scale just described. For example, we have the octave and fifth defined by the ratios $2:1$, $3:2$, the octave appearing as the second harmonic, and the fifth as the third; the ratio of the third harmonic to the second, forming the interval of the fifth, or $3:2$. Going higher we meet with the major third $\frac{4}{3}$, the ratio of the fifth harmonic to the fourth. Higher still we get $\frac{5}{4}$, the minor third, the ratio of the sixth harmonic to the fifth, and so on. The harmonic scale, as obtained from a vibrating string, contains all the pleasing intervals, and when discords occur they can be explained by the action of beats, which are formed when two notes or their overtones send waves of sound to the ear that confuse it and produce in it a sensation which, although not always disagreeable, is at least discordant.

Let us, for a few moments, direct our attention to a question which is intimately connected with music, and which, at the same time, is often misleading. Has colour any connection with music? If we look closely into the matter, passing over the more attractive but visionary theories, we are forced to admit that colours and musical notes have but a forced relation. We may, perhaps, with some degree of accuracy, compare the monotonous tone of a tuning fork with the simple sensation produced by monochromatic light. We might even compare, with a little straining of the imagination, white light which we know

to be compound with the musical sound from a stretched string. But to compare a scale in music which is essentially a series of steps, a discontinuous arrangement, to a spectrum obtained from white light is quite erroneous. As some one has said, there is no music to the eye. Colours without form are merely pleasing; and no combination of colours without form can produce the effect on the soul which is produced by a chord in music. We might, perhaps, with advantage, compare music with painting or architecture; but in the end analogies fail and we are forced to admit that all other things are mortal, but music is Divine.

In conclusion, I want to call your attention to an important point which years ago troubled me and which can be easily explained by reference to the harmonic scale. You all know doubtless that the average individual is fond of music of some kind or other, and that instances of absolute lack of appreciation of music or of a musical scale are extremely rare, and where there is no musical ear or fondness for music or musical scales, there must be a deficiency in the ear. In most cases, however, even when the individual has a fondness for music, but has had no training in the observation of musical intervals, there would appear to be a deficiency. For instance, if we strike a note on a stretched string, rather low in pitch, and ask the average person to whistle it, almost invariably the octave above, two octaves above, or a fifth in one of the octaves will be chosen. This is notably the case with children and with men even when they are asked to sing the note. In most instances the fifth is chosen, particularly with men, the octave being chosen by women. Sometimes even we find the fourth chosen, and in a few cases which came under my notice a minor third in the octave above. Now this would be called, by strictly musical people whose ears are trained, singing out of time. But the explanation is simply that the ear, in its natural untrained state, unconsciously

picks from a series of notes which we know exist in a vibrating string, one which is most prominent—usually a fifth, which is often more prominent in intensity than the fundamental itself—and it is only following a law of nature. This happens in all cases where the ear is untrained and two sounds of different *quality* are to be compared, for the same individual who whistles a fifth or a twelfth above the pitch of a stretched string will have no difficulty as a rule in tuning two strings, which have the same quality, in unison.

When, therefore, one accuses a person of singing out of tune, let the accuser be sure that the note chosen is not one of the more prominent harmonics; and on the other hand, let those who do not easily recognize or separate musical intervals, remember that in nine cases out of ten the fault is not in their ears, but in themselves.

And in closing, to those of you who are fond of mathematics or scientific pursuits, who have a natural love for numbers, I would say cultivate a taste for music. In the theory of music you will find ample scope for your natural bent of mind, and you will be amply repaid. To those of you who love music but not mathematics, I would say nothing; but to those of you who love both music and mathematics, and know a little Latin, I can do no better than quote the words which Cicero used in reference to literature, but which, if he had known better, he would have applied to music and mathematics.

“Nam ceteræ neque temporum sunt, neque ætatum omnium neque locorum; hæc studia adolescentiam agunt, senectutem oblectant, secundas res ornant, adversis perfugium ac solatium præbent; delectant domi, non impediunt foris, pernoctant nobiscum, peregrinantur, rusticantur.”

CICERO, *Pro Archia Poeta*, Sec. 7

THE RELIGION OF ALGEBRAIC CURVES;

MATHEMATICS APPLIED BY ANALOGY TO RELIGIOUS
BELIEFS.

"Scepticism + Credulity = 0."

BY PROF. I. E. MARTIN, B.A.

MANY sceptics argue that religion should be examined by the scientific method, and that whatever cannot stand the hard test of reasoning and evidence should be rejected. They attach little or no weight to the emotional sensations, which form so large a share of modern religion, and which are dependent, to some extent, on a man's health, as well as upon his ideas of morality formed from his manner of living. Champions of orthodox religion unfortunately appeal to personal experience, repudiating any mathematical reasoning as being inapplicable; but this must be done through an entire misconception of the nature of a mathematical law.

We hope to shew that many notions in the religious world are quite analagous to those in mathematics; and that mathematical law, correctly conceived, is quite favourable to orthodoxy, which need not appeal to the emotions, but may argue on strictly scientific lines. The reverent believer shrinks from the thought of the Divine Spirit being bound by mathematical law; but even when tried by this unimpassioned abstraction of science, the theory of a Personal and (speaking in all reverence) interfering God is not only possible, but in the highest degree probable and scientific.

Orthodox believers, accepting the application of fixed laws to the physical world, say that a Personal God, who is the originator of all law, must be above the law; and that mathematical reasoning is inadmissible in discussing the relations between the moral and the physical world, as it would limit the power of the Deity. At once the sceptic asks: "If the physical universe is governed by scientific and, therefore, mathematical laws, how can these laws, in the remotest degree, be altered or affected for your special benefit by prayer to a Personal God?"

Now a mathematical law may be defined to be merely a statement of equality, or, in other words, an equation; and the above popular belief that a mathematical law must be indicated by a regular, inelastic, unaltering, continuous series of effects due to certain causes is quite erroneous. Indeed, such a law may, and often does, within certain limits, represent the most irregular, elastic, variable, and discontinuous effects; and only by supposing the universe to be directed by such laws is it possible to conceive (but what is human conception?) even an Infinite Intelligence having retained infinite latitude to deduce absolutely the most varying, contradictory and incongruous effects from invariable and constant causes, such as seem to be the moving agents in our system of the universe.

To repeat, every law without exception, even the most subtle of the physical universe is mathematically governed, and may thus be conceived as expressed by equations and curves. But, in the first place, these laws must be directed, and we must have a method of calculation which can overcome the immense technical mathematical difficulty in which the simplest physical problems are embedded. Thus, such a simple problem as the continuous flow of water has, for want of a sufficiently good method of calculation, baffled the highest mathematical skill. Many physical problems are capable of solution, but by far the greater number of them even when general laws regarding them

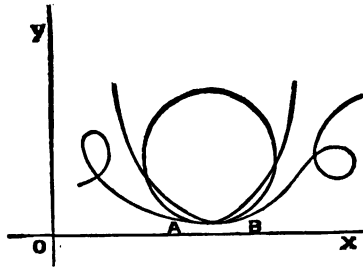
have been ascertained, cannot be explained or disentangled by the best method of calculation yet devised by man. Mathematical science is still in its infancy, and at a future date it hopes to be able to explain, and possibly to control, physical phenomena.

In spite of our only moderate success in practically solving natural problems, we have learned from our successful solutions that the most startling irregularity in the graphic representations of a law or equation may be compatible with the strictest adherence to scientific order. Hence we conclude that no apparent change in the mathematically governed phenomenon can be scientifically pronounced, from observation of what has gone before, to be an arbitrary act of will or a violation of law, unless the complete law of the phenomenon is known from its equation, *i. e.*, unless its curve, speaking technically, is completely traceable from positive to negative infinity—from the infinity of the past to the infinity of the future.

An equation is called of the n th degree if the power of one of the unknowns x , y , or z , is the n th. The unknown coordinate (x , say) with that power may have n values, and thus if $n - 1$ of them are substituted for x^{n-1} in turn we get n different branches of the curve. Thus for any particular phenomenon the mathematician can imagine a law, and, clothing it in an equation, can shew it graphically by one out of the n equally probable fantastic branches of a curve whose degree is the n th.

Again, simply by instrumental observations of the motion of bodies, and by computations resting upon them, the orbit in which the bodies are moving can be ascertained. It was thus that Kepler learned that the planets revolve in ellipses about the sun, before Newton discovered the law of Universal Gravitation, using which, it can be demonstrated that these bodies must so move. But in default of sufficiently powerful means of observation, small arcs of many dissimilar curves may agree, and so to determine the

exact curve in which a body is moving, or the data requisite for the equation of a certain phenomenon it may require a long lapse of time, possibly centuries, for the movement or data or means of observation to be sufficiently developed. Even then we cannot be certain that we have acquired everything necessary to the complete expression of the equation shewing the effects. Thus, mathematically speaking, all mere reasoning from past observations taken probably with imperfect means, and all mere appeal to past experience of natural phenomena is as unscientific as the assumption that because a portion AB of a curve coincides as closely as human observation can appreciate with a portion of a finite circle, therefore a body moving



along an erratic infinite curve, is moving along a finite circle or an infinite parabola which practically coincides with thirty degrees of the circular arc. The movement over the arc AB may have taken years, centuries, or even ages; and another source of error lies in the fact that the methods of observation in the first part of the arc are less accurate than those in the latter part, and therefore not as reliable.

The overwhelming majority of physical problems have not yet been defined by any certain mathematical lines. Take for example, the well known law of gravitation, and suppose it represented by the circle ABD in the last figure. An equation of a curve of which a certain number of points have been accurately determined, can

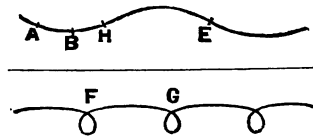
always be found; but a portion AB of an irregular curve may be so nearly circular that the difference cannot be detected by human observation. Now our actual observations on gravitation may perhaps have extended as to time and space over an infinitesimal portion of the whole curve which extends from the infinite past to the infinite future, and therefore as the erratic curve, above mentioned, was not necessarily a circle, so it would be mathematically incorrect to argue that a violation of law is necessarily involved in any irregularity in the movement of bodies governed by gravitation. We know that all astronomical and other observations are at least but fair approximations, since the most infinitesimal deviation of the plumbline (whether due to local attractions, or to those earth tremors, which, like daily tides, seem to sweep over the earth's surface), would cause enormous differences between the real and apparent positions of the heavenly bodies. It is quite allowable to calculate on such data as we possess for the present, but not for the infinite future.

Again, sceptics say that reasoning from the past experience of modern days miracles are special interventions, and are impossible. We shall now indicate the great weakness from a mathematical point of view, of such an argument.

Curves, especially those delineating a mathematical law, and which have a complicated or a transcendental equation of some higher order, often have points where the curve "goes back on itself," becomes discontinuous, passes from concave to convex, returns in loops, or otherwise presents an abnormal character. These points, in the graphic delineation of the law may aptly be called the "miracle points," as the curve suddenly seems to develop at variance with the law, though actually growing in strict accordance with it.

Two distinct processes have been invented for tracing a curve from its equation: first, the purely algebraic;

second, the method of the differential calculus. In the former a curve, or probably only a portion of it, can be traced from points found by deducing values of one coordinate from assumed successive values of the other. When the equation is above the second degree, this process is generally one of approximation, and when the equation is a complicated or transcendental one of a high order, it is so laborious as to be practically useless except to determine a very few points of probably an infinite curve. Moreover, this purely algebraic method is quite powerless to detect, except by the merest chance, the "miracle points." Confining ourselves to coplanar geometry, we have the two curves shewn in the accompanying figure forming the positive and negative branches of a single equation.



We might, perhaps, be able to trace parts of the branches such as A to B , or C to D , by points, but we could not find the exact miracle points H , E , F , G , where the positive branch passes from concave to convex, and where the negative branch returns upon itself in loops. To trace the remaining parts the differential calculus must be employed, which, though not an algebraic process, makes use of algebra. Therefore, before it can be asserted, with any show of mathematical proof, that a so-called miracle is impossible or contrary to law, we must have three things with respect to the moral and physical phenomena bearing on the miracle.

1st. The whole and exact law of the phenomena, and not merely an approximate one from incorrect observations.

2nd. The power (corresponding to the algebraic process of tracing by points) of fixing the normal course of that

law at known points, in other words, under known circumstances.

3rd. The power (corresponding to the process of differentiation) of detecting the unknown points or unknown circumstances, at which or under which the absence of the miracle would actually be a violation of the law.

Now, no moral or natural philosopher can, for an instant, claim to be possessed of all, or even of one, of these laws or powers. Quite certain it is that we are acquainted with comparatively few of the physical forces at work in the universe, and even though at some future day these may all be traced to one centre of force, yet at the present time we have no law regarding their correlation which can be practically applied to some of the simplest physical problems. Certain facts are known, but not the reasons. Thus we know that a trace of gold in steel acts like tempering in hot oil by increasing the tensile strength, but why one acts like the other, or why either one has this peculiar effect is a mystery. This is only a physical phenomenon, but when we come to those cases where mental and physical causes act simultaneously, as is illustrated by mad people who, though naturally weak, perform wonderful feats in strength and agility, apparently drawing enormous mechanical power from no appreciable source, all the data for scientific calculation are absolutely wanting.

In our figure let the positive branch be considered the physical and the negative branch the moral side of the phenomenon delineated by the curve. Then just as we can roughly guess at a small portion *B* to *H* of the positive branch if we know the shape of the part *A* to *B* just in front of it, so may the physical aspect of any phenomenon, for want of better data, be fairly and profitably (though only roughly) discussed by the light of past experience. Again, we see that by looking at the shape of the positive branch alone we cannot conclude the shape of the

negative branch ; and, quite similarly, by looking at the physical aspect of any event we are unable to even guess at the moral aspect. Nor by looking at the positive or physical branch can we deny the existence of the negative or moral branch ; and yet we cannot doubt that, though dissimilar and separate, the two branches are represented by one equation and embody one and the same physico-moral law.

But the moment we speak of physico-moral laws and hint at the possibility of the moral world being governed quasi-mathematically, the following objection suggests itself, viz. : " Mathematics show by analogy that physical miracles may not be departures from law, but if the moral world is so governed what becomes of the free-will of God and man ? Do we not pass from the extreme of scepticism to that of fatalism ? "

The answer is simple. A reasoning man, governed by physico-moral laws, may be compelled by circumstances to go from one place to another, retaining however complete option as to route ; but an unreasoning electric current, governed by physical laws alone may also be compelled to do the same thing, yet only by forcing it to keep in a defined insulated wire. In one case free-will is exerted, in the other it is not, but in both cases obedience to law or circumstances is unavoidable. In mathematical language, reasoning man is given moral laws expressed by equations with full liberty to select their arbitrary constants ; unreasoning matter is given physical laws expressed by equations whose arbitrary constants have been fixed beforehand.

Analogies may be pushed too far, and so let us give an illustration of arbitrary constants, as this mathematical view bears so strongly upon the question of free-will. We know, for example, that all algebraic equations of the second order represent some sort of conic section,—an ellipse or circle, a parabola, or a hyperbola. A master may give his

workman a complete numerical equation of the second order, such as

$$3x^2 - 4xy + 3y^2 + y - x - 9 = 0$$

and tell him to trace its graphic representation. The workman will determine: 1st, that it is a conic section; 2nd, that it is an ellipse; and 3rd, that its major and minor axes are 4 and $\frac{4}{\sqrt{5}}$ respectively, and so on. No free-will whatever is left to the workman. In order to satisfy the master's law he must trace that particular curve and no other. But now let the master give him the general equation of the second degree

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

and tell him to trace the resulting curve, giving him, however, full liberty in selecting the arbitrary constants a , b , c , etc. He finds as before that his free-will is directed to the extent of tracing a conic section of some sort, but that, exercising his free-will in assigning values to the arbitrary constants, he may trace any conic section of any possible size and shape, still completely satisfying the master's law.

The physical forces of the universe may be represented by the numerical equation expressing general fixed circumstances, and the moral forces by general equations also involving general laws, but modified in endless variety by the free-will of reasoning man.

To assert that the moral world is linked to the physical world by some such bonds as connect the negative with the positive branch of the same algebraic curve is not such a wild flight of imagination as is indulged in by modern philosophers who trace back all life to a protoplasm to meteoric dust or cosmic matter, or to a problematical moss-grown stone from another planet. Surely a minority of men who can thus far indulge their fancies, cannot tax with credulity those who, guided by mathematical analogy, believe that as every point in the negative branch of our

curve (above) has its corresponding point in the apparently disconnected positive branch, so every prayer has, by law, its echo in the physical world. Just as cusps in the positive so often correspond to multiple points in the negative branch, so special physical miracles may, by law, answer to special conditions of prayer. But however rational such a belief may be, it is not necessary to hold it in order to understand how prayer may fit into and correspond with physical phenomena. The flexibility introduced into moral laws by free-will (*i. e.* the free selection of arbitrary constants) as contrasted with rigidity of physical laws (*i. e.*, unalterably fixed constants) is, in itself sufficient to their harmonious workings, without supposing them to be necessarily linked together by one common law, in virtue of which the flexible curve of one may be made to fit into the fixed curve of the other for as long as we like.

Prayer may be considered from another point of view, and this manner of looking at it will appeal with great force to the numbers to whom the impulse to pray on every conceivable subject, and in every imaginable circumstance, is as natural as is that which draws the child to the mother in its every day life and difficulties.

Let us apply the mathematical analogy. We may suppose that the master tells the workman that though entire freedom is given to select for himself the constants a , b , c , d , etc., yet that if asked the master himself will explain the condition which must exist between them to represent a circle, ellipse, parabola, or hyperbola, still keeping the general condition that the curve must be a conic section. We can suppose, too, that when asked he will further suggest how the constants should be selected so as to produce the special curve most suited to any given case. If the workman either neglects to inquire the conditions, or ignores them when pointed out, he may run the risk of selecting the wrong curve and of coming into collision with some unyielding physical law. The Christian defines to

his conscience the conditions of right and wrong, and by prayer secures the Master's guidance in the smallest details of life. Viewed mathematically the Christian in so acting is more scientific than the sceptic, for he trusts less to chance and thereby more fully acknowledges scientific laws.

But when conscience is mentioned the sceptic is ready to say: "No two consciences agree as to what is right or wrong. It is simply a matter of habit and training, for if inspired with one spirit all consciences would agree."

Now the Master has promised the Christian guidance, and his conscience listens to his teaching never thinking of narrow conditions being imposed, for he knows how easily he may fall, yet believes that a strong hand will somehow put him on his feet again without his knowing, exactly how it is done. While shrinking from laying down the law of right or wrong for other men's consciences he tries to act up to Abraham Lincoln's noble saying: "With malice towards none, and with charity towards all, let each man do the right as God has given him to see the right." This is the Christian's unscientific view of conscience.

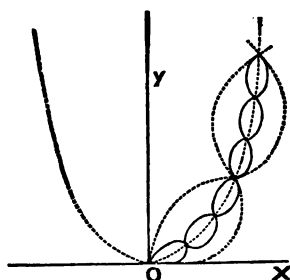
The difference in men's ideas of right and wrong, may also be due to the Master's guidance in the choice of constants. But if we be challenged to take a purely mathematical and scientific view of the matter, the analogies of envelopes and diametral curves will illustrate how conscientious views in the present life may, by the action of free-will, vary indefinitely without infringing the general laws laid down for man's guidance; and also have such diverging views may, in the life to come, all merge into the perfect will of God.

Envelope and diametral curves, indeed, present endless analogies bearing on this view. We shall not take an example of envelope curves, as fresh technicalities would be required; but let us consider the following example of a diametral curve, taken almost at haphazard. It will

shew to a mathematician why is so untenable the theory that because not all consciences agree, they cannot be all inspired by one Master, but merely depend on habit and training. The equation to the curve chosen, is

$$y = ax^2 + c\sqrt{x} \cdot \sin bx,$$

in which a , b and c are the arbitrary constants. Now suppose that while laying down the general law expressed in the equation, the master selects the constant a himself, and leaves the workman perfect liberty to choose b and c . The curve then traced is shewn as follows:



When y is positive, the two branches of the curve form loops alternately receding from and approaching a diametral curve (shewn). The equation to this parabolic curve is $y = ax^2$, and when $\sin bx = 0$, the branches of the curve meet it. When x is negative, y is impossible, except when $\sin bx = 0$, and then $y = ax^2$. So that on the negative side there is an infinity of conjugate points, all situated on the parabola, opposite the double points in the positive branches and with successive abscissae

$$\frac{\pi}{b}, 2 \frac{\pi}{b}, 3 \frac{\pi}{b}, \text{ etc.}$$

The greater the value of b and the smaller the value of c the less do the loops on the positive side diverge from the diametral curve, and the oftener do they intersect upon it. Thus the greater the value of b and the less that of c , the closer do the points on the negative side approach each

other ; and if b were infinitely great, or c infinitely small, *i. e.* $= 0$, they would form a branch of the parabola, whose equation is $y = ax^2$.

Now see the analogies that the trace of this curious curve suggests to the mathematician. Let the present life be the positive, and the life to come the negative side of the curve. The Master wills the general law as expressed by the complete equation, and then by fixing the constant a , He defines His special perfect law from which in the life to come there shall be no divergence, and which in the present life, is the diametral or leading curve. Along this, although not actually a branch of the curve, all consciences must group themselves if they conform to the general universal law. The workman is allowed complete free-will in the selection of the constants b and c , and thus has the utmost latitude of conscience without in any way interfering with the diametral curve. According to this free-will selection (which is doubtless influenced by time, education, locality, etc.), each man's conscience will guide him in a path approaching and receding from the diametral curve in compliance with the general law, and subject only to the condition that in certain cases (corresponding to the double-points of the loops) the law of his conscience, if obeyed, must lead to a point of coincidence or complete accordance with the leading curve of God's perfect law.

The figure illustrates how the different selections of the constants b and c cause different loops to intersect the parabola at totally different points, and how the selection of these constants affects the number of intersections with, and amount of divergence from the diametral curve. If b be very large, and c very small, the loops will almost coincide with the parabola, between the numberless close intersections diverging almost inappreciably from it ; and if b be very small, and c very large, the loops will diverge widely from the parabola, meeting it only at a few distant points.

Precisely in the same way may the course of action of two men in apparently similar circumstances be widely different, while they both follow God's law and obey their consciences. If, however, one man submits himself more completely than the other to the guidance of the Deity, he will (speaking in all reverence) be taught, in answer to prayer, to select proper values for the constants b and c , and in consequence will conform more closely to the will of God, and be in complete harmony with it. The other man, relying on his own knowledge, may unwittingly select such values of b and c , that though obeying the general universal law, he may diverge far from the will of God, and seldom be in complete harmony with it.

As every double point in the positive branch has a corresponding conjugate point in the negative branch, we may, by analogy, suppose that those who, in this life, least diverge from, and most frequently are in complete harmony with the perfect law of God, will, in the life to come, be most like God, the reverse being the case with those who in this life do not live very close to God, or who deliberately ignore or violate the right as God has given them to see the right. From this point of view conscience may be regarded not merely as warning us in this life, but also as registering the influence of every thought, word and deed upon the life to come.

The thought further arises that as the parabola on the positive side is not a part of the curve, but is merely diametral to the actual branches, so complete accordance with God's perfect law may not be attainable, owing to our physical disqualifications, in the present life, yet the more we watch and pray, the closer will it be followed.

ROYAL MILITARY COLLEGE,
Kingston, Dec. 29, 1891.

THE WAVE THEORY OF SOUND.

BY C. A. CHANT, B.A.

A FEW years ago, while living in the northerly part of Ontario, it was my privilege to associate with an elderly gentleman whose judgment in things religious and scientific I respected very highly. Among other books possessed by him was one entitled "The Problem of Human Life," written by an American named Wilford Hall, which he recommended me to read. The book was written with the object of throwing more light upon the great discussion of man's consciousness and the future life, and the number of copies which had left the press was very large. As my tastes did not then strongly incline me to the subject, I did not read much of the work; but I noticed that the fifth and sixth chapters, which comprise over 260 pages of the octavo, were headed, "The Evolution of Sound," and consisted of a long-continued series of impetuous onslaughts upon the generally-accepted theory of sound—*i. e.*, propagation by means of undulations.

More than a year ago I met my friend again, and he was as firmly convinced as ever of the correctness of the position assumed in the book. Some time after that I spoke of the work to a well-known mathematician and physicist of Toronto, and asked him if he had read it. "Yes," he replied, "until the hair stood up on my head." I was more interested now than ever, and determined to examine the book as soon as leisure presented an opportunity; and the present paper is a sketch of my examination.

To any one at all conversant with physical investigations and hypotheses, the task placed before himself by the

author of this monograph on the wave theory is somewhat startling. Of all theories in modern physics, the one that sound is produced by an undulatory mode of motion has always been considered the most firmly fixed. It is not very long since it was supposed by the greatest scientific minds that light and heat were substantial emissions, and in the case of these phenomena we have to hypothesize the existence of that almost infinitely attenuated ether which is supposed to freely circulate between the molecules of the densest matter, and which, the physicist believes, fills up the vast immensities of space. What our brain announces as light, is very probably, outside of the eye, only quiverings or undulations of this substance. But in the case of sound, which (the author says) was declared by Pythagoras and every one that followed to travel by means of waves of the conducting substance, we have the very material which is necessary for its transmission right within our grasp. I think it cannot be denied that the hypothesis of the luminiferous ether has required so many sub-hypotheses to assist it in special cases, that much of the simple symmetry of any great principle is gone, and we very much need another Fresnel to arise who can take the heterogeneous mass of suppositions already put forward, and in their place enunciate some simpler theory capable of explaining the various results reached by experiment. But in the case of sound, what could be simpler than to suppose that the tone of a tuning fork at a distance is transmitted to the listener's ear by means of vibrations of the intervening air? Nevertheless, our author starts out with the promise: "I now come to the work of argument and proof, and shall endeavour to satisfy the reader that the current and universally accepted wave theory of sound is demonstrably a pure and simple fallacy of science, founded upon the most superficial misapprehension of nature and her laws."

The review certainly is remarkable. In some places it

is really ingenious, and I think the student of acoustics would find much profit in considering the explanations given to many even standard experiments in sound, and the objections raised to those usually put forward. Another rather remarkable point in the work is the spirit in which it is written. All through the book we have the strongest expressions of mighty self-confidence, and the writer does not forget to insist, in fully two score of places, that "The wave theory is a most transparent and unmitigated scientific fallacy" (p. 75, etc.); and that the proposition of waves is a "pure fallacy of science without one fact or, when fully analysed, appearance of fact, to warrant them—as will fully appear in due time." (p. 75.)

Again, we are somewhat wearied with the bombastic self-assurance and ostentatious bluster indulged in when the author introduces to us his "star" performer, the little locust, as follows: "I now proceed to present a single argument, which, in its ramifications and various phases will form an avalanche of testimony against the wave theory so overwhelming that its strongest advocates will be forced to recognise it as entirely unassailable." (p. 129.)

This and many similar instances of an overweening self-reliance, together with almost numberless extravagant flourishes over the defeated and cowering heads of Professors Helmholtz and Tyndall, are quite annoying to the thoughtful and honest believer in the current wave theory, who does not care to renounce that belief except for well attested scientific facts—not just for a superabundance of empty pretension, even though expressed with a most fluent verbosity. And moreover this would hardly be expected by many of those "who may conscientiously desire to arrive at the truth, and nothing but the truth," to whom the pages of the volume are "affectionately inscribed." The two chapters on the nature of sound are long, and I shall not attempt to examine all of the arguments produced; but with the assistance of a few sharp-

pointed facts and clear-cutting principles gathered in the Physical course, I hope to prick several of the bubbles which this mighty prophet of extirpation has seen fit to blow.

But first a word on the general method of argument. After assuring the reader most confidently what he will certainly do, the author will take up some experiment or some statement in the works of Tyndall and Helmholtz, and after a very brief examination he will give what he thinks should happen, and call at the same time upon the waiting scientific mind to witness the great impossibilities involved on the face of his opponent's statement. Then, simply on his own assumption as foundation, he will proceed to build some mighty castle so that the whole appearance of the other statement may be ridiculous in the eyes of him who reads. But all that is required is to deny his assumption and the whole house collapses. This trick is resorted to repeatedly; and, indeed, to the uninitiated the statement that two sounds can produce silence, or that, in a single second, seven hundred millions of millions of waves of violet light enter the eye—to the unscientific reader, I say, such statements as these appear but emanations from an unbalanced mind.

But, then, there are several places where there is exhibited a surprising ignorance of some of the very fundamental principles of mathematics and physics.

His conception of a right line is indeed very crude and altogether peculiar. Having shewn that the low E of the double bass has a wave-length of twenty-eight feet in air, and of 476 feet in iron he proceeds to discuss the amplitude. He seems to imagine that it must be transverse to the direction of propagation—as in a violin string—and asks us to consider what the amplitude in this particular case is. Then he asserts that because we cannot notice, with a microscope, this sideways swing in a bar of iron there is none. Further, he inquires:—"If a line were drawn 476

feet in length, so nearly straight that a powerful microscope could not reveal the least deflexion, is there a mathematician on earth who would not, without a moment's hesitation, pronounce that a right line"? (p 245.) If there is one, he had better take the second moment to consider the matter. I am sure he would decide that no line is straight unless it is absolutely and rigidly so. Euclid does not seem to have dreamt of a microscope when he defined a straight line. Now this is one of the basal ideas in mathematics, but the misconception here given is by no means the only one to be found.

We can always excuse a general reader for having an erroneous conception of the meaning of some technical scientific term, especially if that term is used in a wider sense in common conversation; but when an investigator rises to shew that though he is not supplied with "the powerful intellectual abilities of such physicists as Helmholtz, Tyndall, Kundt, Blaserna, Mayer, and a host of others," yet that these men in all their investigations on sound are "astonishingly superficial," and that there cannot be found "such a stupendous philosophical monstrosity and at the same time such a pitiable fallacy as this undulatory theory of sound,"—notwithstanding the fact that these same able minds have carefully and conscientiously spent years of assiduous toil in the elaboration and verification of physical theories—when such a man comes forward with such pretensions and we find him stumbling over simple definitions, we must only regard it, to use his own words, as an "unmitigated philosophical blunder." Such a mistake is made in considering the term *elasticity*. He searched through all the dictionaries and then reached the conclusion that lead is "almost entirely devoid of elasticity," and that pure gold and copper are "likewise practically devoid of elasticity." If we bend a piece of wire into a spiral the author would acknowledge that it has considerable elasticity, but if the bare wire were taken

instead of the spiral, he would find that to make a very small extension a great force is required, and any elementary scientific work would have explained the true meaning of elasticity; that it is measured by the force, resisting extension, or the force with which the displaced particles tend to revert to their original position, which force is equivalent to that producing the change.

Another of these blunders is in the author's use of the term *pressure*. He is very anxious to find the relation between pressure and temperature in an adiabatic compression of air, and for this coveted information he searched through a number of books on pneumatics, but all in vain. It did not occur to him to examine some good work on thermodynamics. Thus he was compelled to construct apparatus to determine the required relation. This was simply a strong glass tube, with a closely-fitting piston. Dropping a thermometer into the tube, the air is compressed to half its volume, and thus the pressure is doubled. Though the writer acknowledges that its intensity will be that of two atmospheres, yet he is of the opinion that the pressure on the thermometer is only fifteen pds. to the square inch—a single atmosphere. Thus he finds the mercury rises $2\frac{1}{2}^{\circ}$, and allowing for radiation an amount of heat equal to that still retained, he considers that for a truly adiabatic compression, the rise would be 5° . But he is assuredly assuming too much in stating that for a rise of fifteen times as much, the pressure must be fifteen times as great. He then states this problem: "If a *cubic inch* of air requires fifteen pds. pressure to raise its temperature by five degrees, how much pressure will be required to raise the temperature of the same cubic inch 87° ?" He figures it thus:

$$\frac{87}{5} \times 15 = 255 \text{ pds. nearly (p. 144).}$$

and triumphantly exclaims "Our experiment demonstrates 255 pds. of pressure to the cubic inch is the mechanical

force necessary to generate the required 87° of heat," (p. 145). The conceptions of force, work, heat and temperature here involved are very loosely handled; but who ever heard of *pressure per cubic inch*? And yet witness the brazen assurance in the reference to Professor Tyndall, just following: "The professor ought to thank the writer for correcting this important defect in his book, and for having furnished him the proper scientific data for his next course of lectures on sound," (p. 145).

But, perhaps the most glaring exhibition of rank ignorance of the simplest notions underlying every branch of mathematical science is in the author's juggling with units. He accuses Professor Tyndall of having "consciously perpetrated one of the most glaring and astonishing scientific errors on record" (p. 163), but says "that it will take but a few remarks to expose and correct this fundamental error in science, assumed as it is in all works of natural philosophy, and thus shew the reader what kind of scientific information is being sown broadcast through the land for the enlightenment of college students," (p. 163).

Professor Tyndall states that the intensity of sound varies inversely as the square of the distance, and to explain this law he takes one foot as the unit of length, saying that at the distance of three or four feet from the sounding centre the intensity is $\frac{1}{9}$ or $\frac{1}{16}$ respectively, of what it is at unit distance. The reviewer argues as follows:—A mile is 5,280 feet and a steam signal service siren can be heard for ten miles, or 52,800 feet; hence at the distance of ten miles the sound will possess but $1/2,000,000,000$ th of the intensity *at the start*, (p. 152), and he continues: "Had he employed rods as his measure for determining the decrease in loudness as the square of the distance from the sounding body, in the place of feet, we would find the sound of the steam siren at a distance of ten miles, to be diminished in loudness only $1/10,000,000$ th, instead of $1/2,000,000,000$ th, as we get in employing a foot

as our measure, and would thus have approached 200 times nearer the truth, since the supposition of any sound being distinctly audible after being reduced to $1/2,000,000$ th of its NORMAL INTENSITY (!) is so preposterous that it only needs to be stated to be refuted," (p. 163). Here we can see the style of argument—*argumentum ad hominem* and fundamental bungling. Again, "if we had used inches, the writer says the sound would actually possess" but $1/400,000,000,000$ th as much intensity as *at the start*, or, in other words, is $1/400$ th as loud as it would be by adopting feet as our measure. Then half a mile is chosen as the unit with the result that the intensity is only $1/400$ th "of the intensity *at the start*." To dispel the darkness which he has thrown about us, he states the simple rule to be employed, which he considers is a truism. It is: "The measure to be employed in computing such proportional decrease in the intensity of particular sounds, if we estimate by the square of the distance at all, must always, and of necessity, vary exactly in proportion to the intensity of the different sounds at the start, or, in other words, as the range of the different sounds varies," (p. 164). He adds: "Of course Prof. Tyndall never thought of this," and I am sure we all fervently hope that he never did.

Considering these few evident mistakes—and there are many more of them—but simply looking at these enumerated, I think I am justified in using his own words, as follows: "I now appeal to the logical intelligence of the readers of this monograph, if it is possible for a theory to be based upon scientific principles which ignores such simple truths," (p. 276); and, moreover, the author, unless he sees fit to correct some of these "transparent fallacies," will "justly earn and should receive the ridicule of the whole scientific world," (p. 263.)

The first real attempt at argumentation is to be found in the discussion of sympathetic vibrations and resonators. As is well known the cause of this sympathetic vibration

is said to be the dashing of air waves against the body whose normal time of vibration is the same, or very nearly the same, as that of the body sending off the waves. To shew the incorrectness of these views, three cases are given in which, as the writer states, the theory breaks down. If two isochronous pendulums be suspended near each other, he asserts that one of them in motion does not produce any visible effect on the other. Also if two sonometer strings be stretched from rigid iron supports, or if two unison tuning forks be dismounted from their resonant cases, it is said there is no sympathetic action between them. In the first instance the slowness of vibration and the mobility of the air allows the air in front to easily escape from before the advancing pendulum and pass around behind it. In the other two instances, I believe the author is in error; that there is a sympathetic action. Again it is asked how a little string can place in motion a sounding-board whose mass is a thousand-fold greater. This idea is scouted as preposterous; but, notwithstanding, I believe it is quite true, and quite as easy to conceive as that one man by pulling at a rope can lift a ton. The string is the commanding officer, the sounding-board is the army awaiting orders; or, the string is the piston, the board is the balance wheel.

A rather more interesting case is that of the relative merits of a "pine chip" and a piece of iron. The writer asserts that if a pine chip be held touching the lower end of a tuning-fork, a considerable reinforcement of sound is heard; but if the pine chip be replaced by an "iron chip," similar in size and shape, this reinforcement is absent. In this case the reason of the superiority of the wood must depend upon its molecular structure.

There are certainly many novel examples of reasoning in the work, and one of the newest to me is the attempt to shew that the pendulum above mentioned should be more potent than a tuning-fork in carving undulations.

Let us consider a fork whose vibration-frequency is 128 to the second. The excursion of the prong is but, say $\frac{1}{8}$ in. from the position of equilibrium, or $\frac{1}{4}$ in. between the extreme points of the motion. Now the distance travelled in one direction by this fork's prong, will be only $128 \times \frac{1}{4}$ or 32 inches per second; and it is asserted that this velocity is not great enough to produce any condensation in the free air; and that, therefore, there is no such thing as an air-wave. Along with the demolition of the air-wave goes, of course, the annihilation of the wave-theory of sound. As another proof that the air-waves are imaginary, he went into a closed room, lighted a gas-jet at one end, and then at the other vigorously waved a rigid fan, and then, "to the utter discomfiture of the wave-theory, the experiment shewed that the delicate and sensitive gas-jet could not be stirred at a distance of more than twenty or twenty-five feet, while it took the most powerful waves I could produce, five seconds to travel the distance." We are somewhat amused at the crudeness of the experiment, but we fail to see a close analogy between the disturbance caused by the fan and the supposed air condensations and rarefactions. I imagine if there were a bald head between the fan and the gas flame, a cool breeze would be experienced passing over it. This was a wind, not a transmission of air-waves in the true sense.

But having thus demolished this ancient theory by a simple motion of his "magic fan," let us see what he will substitute in its place. The law governing the generation of tone by a vibrating fork or string, he states concisely as follows:—"It is not the numerous short motions back and forth on the surrounding air which generates the tone of a fork, or string, but it is the molecular effect of the *sudden* stops and starts on the atomic structure of the instrument itself, causing thereby a motion of the substantial pulses we call sound while the atmosphere, wood or iron, through which they pass, is but their conducting

medium." That is, simply, that sound is a substantial emission, that it was stowed away in the body and, on being called for, was projected forward at the rate of eleven hundred feet a second.

There are many experiments and arguments somewhat similar to those referred to, but I have not time to consider them all. Yet I wish to outline what the reviewer believes to be the very king of experiments, that, in fact, compared to those already spoken of, it is as a devastating cyclone to a gentle zephyr. I refer to his great "locust story," which occupies about eight or ten pages of the work. (p. 129 and fol.) The little insect which is the hero of the story, and which came to his assistance so fortunately, is one of the *locustidæ*, of the order of *orthoptera*. It weighs only a fraction of a pennyweight, and yet can make its song heard for more than a mile. The sound is produced by the rasping of the animal's legs against its wings. The author quotes Darwin as corroborating his statements, and he says he himself heard one in 1867. The argument is simply this: Within the space where this little insect can be heard is say four cubic miles of air (the volume of a sphere of one mile radius). Our author says there are four *square* miles, and that within this space there are, in round numbers, sixty thousand millions of square inch columns of air, each exerting a pressure on the earth of fifteen pds. Within this mighty limit, affected by the insect, all the particles of air must be subject to the rarefactions and condensations of the waves, and it is the object of the writer to shew that the little locust is entirely unable to do almost the slightest fraction of the work required. He employed two mathematicians to calculate the work required to set in vibration a single inch column of air a mile high; one of them said *fifteen pds. of mechanical force* per second, and the other about forty. I must confess I can see no physical interpretation for the phrase "pounds of mechanical force per second." The

author, to be within the mark, assumes the work to be *one pound per second*; and as there are sixteen thousand million "square inch columns of air," the total work required is sixteen thousand million pounds per second. Now, the little insect cannot stir even half-an-ounce weight, and so the poor wave theory must go. Force, it is always well to understand, can not be measured in pounds per second.

But look at it in another light. In the American Encyclopedia Professor Mayer states that in the compressed part of the wave the ordinary density of the atmosphere is increased by $1/679$ th. Now half of the four cubic miles influenced by the locust is compressed, and the author says that the work done is the same as if the two cubic miles were enclosed in a box and then compressed $\frac{1}{679}$ th part. To double the density of a cubic inch a pressure of 15 pds. is required, and to increase the density by $\frac{1}{679}$ th we must just take $\frac{1}{679}$ th of 15 pds. Now we have simply to find the number of cubic inches in the two cubic miles of condensed air, multiply the result by 15 and divide by 679. This is declared to indicate that there is required the "little physical and mechanical energy of five thousand million tons." Here is another novel mathematical notion—energy measured in tons. Again must the wave-theory retire.

On the evening of the day on which I first read this argument I was standing with a friend upon the large iron bridge which spans the River Ottawa, below the Chaudiere Falls, and while there the bridge was continually vibrating. I saw a light carriage pass over, and in the centre of the span the oscillation was very noticeable. Now this bridge weighs thousands of tons. I might reason that it is impossible that the carriage could have produced the motion as it weighs only a few pounds. Again, if we take a steel rod about $2\frac{1}{2}$ feet long and with a cross section of $\frac{1}{2}$ a sq. inch and clamp it in the middle we can cause extensions and contractions in the bar simply by rubbing

one-half of it with a piece of chamois covered with powdered resin. With a few rubs a quite large extension is produced. To extend by $\frac{1}{4}$ of an inch a steel rod of the above size clamped in the middle, by attaching a weight, would require about 100,000 pds. Surely the performer does not weigh that much. But the other end of the bar is elongated and contracted without being touched! That must surely indicate that there are spirits at work around this end of the bar.

Some of the conclusions remind one very much of the old gentleman who declared that forty feet of water had fallen during a season, as his water barrels were five feet high, and had been full eight times. Surely it is not difficult to conceive that the air is so constituted that one mass can easily transmit motion to the neighbouring particles, and that if this vibration continues for some time, the successive effects add to each other until at last tremendous results are reached.

Besides "over-tones," our author has a theory of "over-smells," and "over-tastes," but I must forego the pleasure of delineating it.

All students of acoustics know that the bulwark of their science is the phenomenon of interference, and to overcome the wave-theory the assault upon this foundation must be backed up by the strongest arguments that experiment and logic can supply. These are not found here.

Helmholtz's Double Sirene consists essentially of two perforated discs rotating at the ends of two similarly perforated wind-chests. Now when the puffs from the upper chest take place at the same time as those from the lower we hear an intense sound; when they take place between those from the lower chest there is a considerable diminution in the intensity, and the tone is that of the first harmonic. The effects are easily explained by the undulatory theory. When the puffs are synchronous, the condensation

of each reaches the ear at the same time, and the intensity is a maximum; but when the puffs alternate, condensation from one is superposed on rarefaction from the other, and we have minimum intensity from the fundamentals, while the harmonic comes out. Yet, the author has a very ingenious explanation. As the puffs now follow each other at intervals only half as great as before; the resulting sound should be the first harmonic—and of the same intensity; and it is just there that the difficulty arises—the resulting harmonic is not of equal intensity. Hence, to use his own words, his explanation is “purely visionary, and monstrously chimerical.”

What always seemed to me the most conclusive and best experiment to verify the interference of sound was that performed with Koenig's apparatus. A tuning fork is placed at the end of a tube, and, by its vibration, transmits condensations and rarefactions to the air within the tube. At a short distance from the fork the tube is divided into two branches, which unite again, however, soon after. If now one of these branches is drawn out to be half a wave-length longer than the other, at the point where the tubes reunite a condensation in one tube should coincide with a rarefaction in the other, and so when one is superposed on the other the effect should be neither condensation nor rarefaction but complete quiescence. Now the author of this work knew what a strong witness this experiment would be for the wave-theory, for he remarks: “It was plainly evident to my mind if the apparatus and its acoustical effects were correctly described, they would strongly favour the wave-theory, and would present an almost conclusive evidence in favour of the law of interference between sound waves as claimed by advocates of the hypothesis.” So to test the correctness of the experimental results he borrowed from a friend the instrument, and after trying his experiments the writer says: “I here declare to the reader and to the scientific world that no such

thing as silence occurs, nor even a respectable approach to it," (p. 309). He does not give any details of the work, though he does state a few odd results reached, each of which, however, is quite satisfactorily explained by the undulatory theory. The best way to convince one's self is to try the experiment, and it will be done on the apparatus before you.—[Experiment performed, result as above indicated.]

There are many other odd explanations and peculiar objections made, but I have not time to refer to them.

Before I read the book I was a firm believer in the wave-theory, but if it were demonstrated false I was perfectly willing to cast it overboard; after reading it, I was still of the opinion that the various phenomena of sound are best explained by the old hypothesis, and that when Cowper states that

the sound of cheerful bells
Just *undulates* upon the listening ear,

he utters a scientific as well as a poetic fact.

SOME TRIGONOMETRICAL EXPANSIONS.

BY W. GILLESPIE.

Expanding $\cos n\theta$ in terms of the powers of $\sin^2\theta$, suppose we obtain

$$\cos n\theta = 1 + A_2 \sin^2\theta + A_4 \sin^4\theta + \dots + A_n \sin^n\theta \dots (1).$$

The first is seen to be unity by making $\theta = 0$.

Change θ into $\theta + h$, and $\cos n\theta$ becomes

$$\cos n\theta \cos nh - \sin n\theta \sin nh = \cos n\theta - nh \sin n\theta - \frac{n^2 h^2}{2} \cos n\theta + \dots (2)$$

Making the same change in other side

$$A_{2r} \sin^{2r}\theta = A_{2r} \left(\sin^2\theta + h \cos^2\theta - \frac{h^2}{2} \sin^2\theta - \dots \right)^{2r},$$

The coefficient of h^2 in this is

$$A_{2r} \left\{ \frac{2r(2r-1)}{1 \cdot 2} \sin^{2r-2}\theta \cos^2\theta - r \sin^{2r}\theta \right\}.$$

Equate coefficients of h^2 ,

$$\begin{aligned} -\frac{n^2}{2} \cos n\theta &= A_2 (\cos^2\theta - \sin^2\theta) \\ &+ A_4 (2 \cdot 3 \sin^2\theta \cos^2\theta - 2 \sin^4\theta) + \dots \\ \dots + A_{2r} &\left\{ \frac{2r(2r-1)}{1 \cdot 2} \sin^{2r-2}\theta \cos^2\theta - r \sin^{2r}\theta \right\} \end{aligned}$$

Changing $\cos^2\theta$ into $1 - \sin^2\theta$ and equating coefficients of $\sin^{2r}\theta$,

$$-\frac{n^2}{2} A_{2r} = -A_{2r} \left\{ \frac{2r(2r-1)}{1 \cdot 2} + r \right\} + A_{2r+2} \frac{(2r+2)(2r+1)}{1 \cdot 2};$$

$\frac{n^2}{2} A_{2r} = 2r^2 A_{2r} - A_{2r+2} (r+1)(2r+1)$, hence we have

$$A_2, A_4, \text{ etc.}$$

$$\cos n\theta = 1 - \frac{n^2}{2} \sin^2 \theta + \frac{n^2(n^2 - 2^2)}{4} \sin^4 \theta + \frac{n^2(n^2 - 2^2)(n^2 - 4^2)}{6} \sin^6 \theta + \dots + k \sin^n \theta \dots (3)$$

To find k , we have, (Todhunter's Trig. p. 229),

$$2 \cos n\theta = (2 \cos \theta)^n - n(2 \cos \theta)^{n-2} + \dots$$

$$\text{or } \cos n\theta = 2^{n-1} \cos^n \theta - \dots$$

Hence we see $k = 2^{n-1}(-1)^{\frac{n}{2}}$, and $\therefore$ (3) becomes,

$$(-1)^{\frac{n}{2}} 2^{n-1} \sin^n \theta \dots + \frac{n^2(n^2 - 2^2)}{4} \sin^4 \theta - \frac{n^2}{2} \sin^2 \theta + 1 = \cos n\theta \dots (3^1)$$

The left-hand side will be satisfied if we make

$$\theta = \frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \dots \text{ since } \cos n\theta \text{ will} = 0.$$

$\therefore \sin \frac{\pi}{2n}, \sin \frac{3\pi}{2n}, \sin \frac{5\pi}{2n}, \dots$ are roots of the equation,

$$(-1)^{\frac{n}{2}} 2^{n-1} \sin^n \theta - \dots - \frac{n^2}{2} \sin^2 \theta + 1 = 0 \dots (4)$$

$$\therefore \text{product of all the roots} = (-1)^{\frac{n}{2}} \frac{1}{2^{n-1}};$$

product of roots taken $(n-1)$ together = 0, since coefficient of $\sin \theta = 0$.

$$\text{Hence } \frac{1}{\sin \frac{\pi}{2n}} + \frac{1}{\sin \frac{3\pi}{2n}} + \frac{1}{\sin \frac{5\pi}{2n}} + \dots = 0$$

$$\text{or cosec } \frac{\pi}{2n} + \text{cosec } \frac{3\pi}{2n} + \text{cosec } \frac{5\pi}{2n} + \dots = 0 \dots (a)$$

Again we can solve equation (4) into roots of the 2nd degree,

$$\sin^2 \frac{\pi}{2n}, \sin^2 \frac{3\pi}{2n}, \sin^2 \frac{5\pi}{2n}, \dots$$

$$\text{Their product, as before,} = (-1)^{\frac{n}{2}} \frac{1}{2^{n-1}};$$

$$\text{products } \left(\frac{n}{2} - 1\right) \text{ at a time} = (-1)^{\frac{n}{2}} \frac{n^2}{2} \cdot \frac{1}{2^{n-1}}.$$

$$\therefore \frac{1}{\sin^2 \frac{\pi}{2n}} + \frac{1}{\sin^2 \frac{3\pi}{2n}} + \frac{1}{\sin^2 \frac{5\pi}{2n}} + \dots = \frac{n^2}{2}$$

$$\text{or } \operatorname{cosec}^2 \frac{\pi}{2n} + \operatorname{cosec}^2 \frac{3\pi}{2n} + \operatorname{cosec}^2 \frac{5\pi}{2n} + \dots = \frac{n^2}{2} \dots (b)$$

and since $\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$,

$$\begin{aligned} & \cot^2 \frac{\pi}{2n} + \cot^2 \frac{3\pi}{2n} + \cot^2 \frac{5\pi}{2n} + \dots \\ &= \frac{n^2}{2} - \frac{n}{2} = \frac{n(n-1)}{2} \dots \dots \dots (c). \end{aligned}$$

The product of the roots, taken $\left(\frac{n}{2} - 2\right)$ together

$$= \frac{(-1)^{\frac{n}{2}} n^2 (n^2 - 2^2)}{2^{n-1} \cdot 4}.$$

Dividing these by product of all the roots we get

$$\begin{aligned} \Sigma \frac{1}{\sin^2 \frac{\pi}{2n} \sin^2 \frac{3\pi}{2n}} &= \frac{n^2 (n^2 - 2^2)}{4} = \Sigma \operatorname{cosec}^2 \frac{\pi}{2n} \operatorname{cosec}^2 \frac{3\pi}{2n} \\ &= \Sigma \left\{ \left(\cot^2 \frac{\pi}{2n} + 1 \right) \left(\cot^2 \frac{3\pi}{2n} + 1 \right) \right\} \\ &= \Sigma \cot^2 \frac{\pi}{2n} \cot^2 \frac{3\pi}{2n} + \left(\frac{n}{2} - 1 \right) \Sigma \cot^2 \frac{\pi}{2n} + \frac{n(n-2)}{8} \dots \\ \Sigma \cot^2 \frac{\pi}{2n} \cot^2 \frac{3\pi}{2n} &= \frac{n^2 (n^2 - 4)}{4} - \frac{n-2}{3} \frac{n(n-1)}{2} + \frac{n(n-2)}{8} \\ &= \frac{n(n-1)(n-2)(n-3)}{4} \dots \dots \dots (d) \end{aligned}$$

We also have (Todhunter's Trig. p. 231,)

$$(-1)^{\frac{n}{2}} \cos n\theta = 1 - \frac{n^2}{2} \cos^2 \theta + \frac{n^2 (n^2 - 2^2)}{4} \cos^4 \theta - \dots$$

and when $\theta = \frac{\pi}{2n}, \frac{3\pi}{2n}, \text{ etc.}, \cos n\theta = 0$.

$\therefore \cos \frac{\pi}{2n}, \cos \frac{3\pi}{2n} \dots$ are roots of the equation

$$\begin{aligned} & (-1)^{\frac{n}{2}} 2^{n-1} \cos n\theta \dots + \frac{n^2 (n^2 - 2^2)}{4} \cos^4 \theta - \frac{n^2}{2} \cos^2 \theta + 1 \\ &= 0 \dots \dots \dots (5) \end{aligned}$$

Products of roots $n - 1$ together = 0, and product of all

$$= \frac{1}{(-1)^{\frac{n}{2}} 2^{n-1}}.$$

$$\therefore \sec \frac{\pi}{2n} + \sec \frac{3\pi}{2n} + \dots = 0 \dots \dots \dots (e)$$

Again, solving for roots of second degree, we get

$$\text{products of all, } \frac{n}{2} \text{ together,} = \frac{1}{(-1)^{\frac{n}{2}} 2^{n-1}}$$

$$\text{products } \frac{n}{2} - 1 \text{ together} = \frac{1}{(-1)^{\frac{n}{2}} 2^{n-1}} \cdot \frac{n^2}{2}$$

$$\therefore \sec^2 \frac{\pi}{2n} + \sec^2 \frac{3\pi}{2n} + \dots + \sec^2 \frac{n-1}{2n} \pi = \frac{n^2}{2} \dots \dots (f)$$

$$\therefore \tan^2 \frac{\pi}{2n} + \tan^2 \frac{3\pi}{2n} + \dots = \frac{n(n-1)}{2} \dots \dots \dots (g)$$

$$\text{Products, taking } \frac{n}{2} - 2 \text{ together} = \frac{1}{(-1)^{\frac{n}{2}} 2^{n-1}} \frac{n^2(n^2-2^2)}{4}.$$

Dividing by product of all we get

$$\begin{aligned} \Sigma \sec^2 \frac{\pi}{2n} \sec^2 \frac{3\pi}{2n} &= \Sigma \left(1 + \tan^2 \frac{\pi}{2n} \right) \left(1 + \tan^2 \frac{3\pi}{2n} \right) \\ &= \Sigma \tan^2 \frac{\pi}{2n} \tan^2 \frac{3\pi}{2n} + \left(\frac{n}{2} - 1 \right) \Sigma \tan^2 \frac{\pi}{2n} \\ &\quad + \frac{n(n-2)}{8} = \frac{n^2(n^2-2^2)}{4}. \end{aligned}$$

$$\begin{aligned} \therefore \Sigma \tan^2 \frac{\pi}{2n} \tan^2 \frac{3\pi}{2n} &= \frac{n(n-1)(n-2)(n-3)}{4} \\ &= \Sigma \cot^2 \frac{\pi}{2n} \cot^2 \frac{3\pi}{2n} \dots \dots \dots (h) \end{aligned}$$

Again, taking the same equation (3¹), by giving θ in succession the values θ , $\theta + \frac{2\pi}{n}$, $\theta + \frac{4\pi}{n}$, $\theta + \frac{6\pi}{n}$, ... we see $\cos n\theta$ remains constant; let it = k .

Hence we see that

$$\sin \theta, \sin \left(\theta + \frac{2\pi}{n} \right), \sin \left(\theta + \frac{4\pi}{n} \right), \dots \text{ are the roots of } (-1)^{\frac{n}{2}} 2^{n-1} \sin^n \theta \dots + \frac{n^2 (n^2 - 2^2)}{[4]} \sin^4 \theta - \frac{n^2}{[2]} \sin^2 \theta + 1 - k = 0.$$

$$\text{As before, product of all roots} = \frac{1 - k}{(-1)^{\frac{n}{2}} 2^{n-1}}.$$

Products taking $n - 1$ together = 0,

$$\therefore \operatorname{cosec} \theta + \operatorname{cosec} \left(\theta + \frac{2\pi}{n} \right) + \operatorname{cosec} \left(\theta + \frac{4\pi}{n} \right) + \dots = 0.$$

or, solving into quadratic roots,

$$\text{products of all, } \sin^2 \theta, \sin^2 \left(\theta + \frac{2\pi}{n} \right), \text{ etc.,} = \frac{1 - k}{(-1)^{\frac{n}{2}} \cdot 2^{n-1}};$$

$$\text{products } \frac{n}{2} - 1 \text{ roots together} = \frac{n^2}{2} \frac{1}{(-1)^{\frac{n}{2}} \cdot 2^{n-1}}$$

$$\therefore \operatorname{cosec}^2 \theta + \operatorname{cosec}^2 \left(\theta + \frac{2\pi}{n} \right) + \operatorname{cosec}^2 \left(\theta + \frac{4\pi}{n} \right) \dots$$

$$= \frac{n^2}{2(1 - k)} = \frac{n^2}{2(1 - \cos n\theta)} \dots \dots \dots (i)$$

$$\therefore \tan^2 \theta + \tan^2 \left(\theta + \frac{2\pi}{n} \right) + \dots = \frac{n^2}{2(1 - \cos n\theta)} - \frac{n}{2} \dots \dots (j)$$

We have also (Todhunter, p. 230)

$$\sin n\theta = n \cos \theta \left\{ \sin \theta - \frac{n^2 - 2^2}{[3]} \sin^3 \theta + \frac{(n^2 - 2^2)(n^2 - 4^2)}{[5]} \sin^5 \theta - \dots \right\}.$$

$$\text{When } \theta = \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \dots \sin n\theta = 0$$

$\therefore$ right hand side of equation = 0.

But we can divide out by $n \sin \theta \cos \theta$, none of which = 0.

∴ the roots of

$$1 - \frac{n^2 - 2^2}{3} + \dots 2^{n-2} \sin^{n-2} \theta = 0$$

are (solving in roots of the second degree)

$$\sin^2 \frac{\pi}{n}, \sin^2 \frac{2\pi}{n}, \sin^2 \frac{3\pi}{n}, \dots$$

$$\text{Product all together} = \frac{1}{2^{n-2}}$$

$$\text{Products, taking all but one at a time} = \frac{n^2 - 2^2}{3 \cdot 2^{n-2}}$$

$$\therefore \operatorname{cosec}^2 \frac{\pi}{n} + \operatorname{cosec}^2 \frac{2\pi}{n} + \dots = \frac{n^2 - 2^2}{6} \dots \dots \dots (k)$$

$$\begin{aligned} & \text{and } \cot^2 \frac{\pi}{n} + \cot^2 \frac{2\pi}{n} + \cot^2 \frac{3\pi}{n} + \dots \\ &= \frac{n^2 - 2^2}{6} - \frac{n - 2}{2} \dots \dots \dots (l). \end{aligned}$$

I could continue, and sum more series by noting that $\sin n\theta$ always is constant when $\theta = \theta + \frac{2\pi}{n}$, or $\theta + \frac{4\pi}{n}$, etc., or by taking another one of the expansions, but I think enough has been done to illustrate the method.

MEASUREMENT OF TIME.

BY G. B. ANDERSON.

IN the early period of the world's history when man's needs were few, the very obvious divisions of sunrise, sunset, and mid-day would probably serve his purpose very well; and if more were required the moon's phases would form a second period; and, lastly, the recurrence of the seasons would mark a period of time sufficiently long for the purposes of primitive races. The practice of reckoning time by moons still survives among nearly all barbaric nations, and even the Mohammedans make their year consist of twelve synodic revolutions of the moon.

The necessity of dividing the comparatively long period of daylight would soon be felt, and what more natural than that the space traversed from day to day by the shadow of some particular and well-known object, be divided into a number of equal parts? This might be done rudely at first on the ground, but soon a mechanical device of some sort would be used, and here we have the sundial, the oldest of all time-measuring instruments. The most ancient of these of which we have any record, is that of King Ahaz, who flourished about 742 B.C. It is stated in Isaiah xxxviii. 8, that the sun went back ten degrees, from which we might infer that the instrument was graduated with some minuteness. Now it was the custom of all the nations of antiquity to have the number of divisions in the period of daylight constant, so that the length of their hours would continually vary with the time of the year, but in tropical countries the difference would not be very noticeable. These were called temporary hours.

About 340 B.C. Berosus, a Chaldean astronomer, constructed a dial consisting of a hemispherical cup placed horizontally, in the centre of which was fixed a small ball which threw a shadow on the inside of the cup as long as the sun remained above the horizon. The curve traced by this shadow being divided into twelve equal parts gave the hour points for that particular day; a number of curves being thus drawn at different periods of the year, and similarly divided, hour lines were then drawn through the corresponding points of each curve, and the dial was complete. This form of dial remained in use among the Arabs up to 900 A.D. Various forms of dials were used by the Greeks and Romans, some of them of considerable complexity, and in more recent times this ancient instrument constructed on mathematical principles did good service where great accuracy was not required, until displaced by better apparatus.

The need of having the time indicated on days when the sun cast no shadow on the dial gave rise to two inventions which made their appearance about 140 B.C.; they were the hour glass and the clepsydra, or water-clock, exactly similar in principle. The former of these needs no description as its essential features are preserved in the modern three-minute glass used for domestic purposes. The first clepsydra is supposed to have been constructed by Ctesibus of Alexandria. It consisted of a small boat floating in a vessel of water with a hole in the bottom, and as the water escaped an oar in the boat pointed to the hours marked on the side or on a central column. We know from the theory of hydrodynamics that in order that the surface of the liquid should descend uniformly the vessel must be in the form of a surface of revolution whose equation is $y^4 = a^3 x$, but of course the hour marks could be determined by trial for any form of vessel. The clepsydra was also constructed by having a tank kept filled with water to a constant level, the water from this escaping to a second

vessel, and since the pressure was constant, the flow would be uniform and the surface of the water in the last vessel would rise uniformly and thus make the hour divisions of equal length.

The water-clock was improved in various ways until it became an indicator of considerable accuracy. The Romans are even said to have constructed it so that the hour of noon was announced by a loud noise, probably produced in some way by hydraulic pressure. Charlemagne received one from the king of the Persians which we are informed struck the hours by allowing a number of brass balls to fall on a metallic plate. Even in more recent times the weight of mercury flowing through a small tube under constant pressure has been used to determine intervals of time. Now although wheel-work was known and applied to machinery in the time of Archimedes, we have no proof that it was ever used in connection with the water-clock, although it is obvious that it might have been so used by a system of floats such as is employed in a wheel barometer, though not perhaps with any great degree of accuracy. These two contrivances, that is, the clepsydra and the hour glass, continued to be used until about the fifteenth century, when clocks driven by weights displaced them. Up to this time the professors and lecturers in the universities carried their hour glasses to the classrooms and when the "sands of time" had run out departed to fresh quarters.

Another ancient device for measuring time, which has been associated chiefly with the name of Alfred the Great, was that of burning candles. King Alfred's candles are said to have been twelve inches long, and to have burned at the rate of three inches an hour.

The time at which clocks moved by weights were invented is exceedingly uncertain, which may be in part accounted for by the fact that the clock was not in all probability the invention of a single individual, but rather a gradual evolution. It appears, however, that clocks of

some kind began to be manufactured in the monasteries of Europe about the eleventh century, and that their invention dated farther back, and was possibly obtained from the Saracens. The first of which we have any clear, authentic record was constructed by Richard Wallingford, abbot of St. Alban's, in 1326; another was put up in Dover castle, in 1348, and this clock was shown in operation at the exhibition of 1876. Henry de Vick constructed a clock with a vibrating balance for Charles V. in 1370. Clocks continued to be made with balances for nearly three hundred years, but their motion must have been more or less irregular, and it was not until the principle of the pendulum was discovered that they could be said to be reliable. To whom is due the honour of this discovery is a vexed question. Galileo is said to have observed that a chandelier hanging from the roof of a church at Florence performed oscillations in equal intervals of time, but evidence is not wanting that the oriental astronomers of antiquity used the pendulum as a time measurer for eclipses and star transits by counting the oscillations. Huyghens was the first to investigate the mathematical properties of it and to show that the pendulum is not strictly isochronous except when vibrating in a cycloid. When this became known the pendulums applied to clocks were made so that they were forced to vibrate in this way by means of cycloidal cheeks interposed at the point of suspension, but it was afterwards pointed out that by this method greater errors were introduced than those which it was sought to counteract. And the pendulum as now applied to clocks is suspended by a thin spring which, since it bends throughout its whole length and not at a single point, causes the vibration to be nearly cycloidal, and since the arc of vibration seldom exceeds four or five degrees the error is negligible. With the application of the balance, and, later on, the pendulum, came the necessity for an escapement of some sort, and various forms of these were in use, each

having its special merits until that of George Graham, commonly known as the deadbeat escapement, was invented which almost entirely superseded all previous forms, and has continued in use ever since. To Graham we are also indebted for the mercurial compensation pendulum, which provides for the error arising from variations in its length due to expansion. It consists of a cylindrical jar of mercury used as a bob, which by its upward expansion raises the centre of gravity sufficiently to counteract the downward expansion of the rod.

Timepieces moved by spiral springs were made as early as the sixteenth century, and of course these are still used in all timekeepers destined to be carried from place to place, under which we include both watches and chronometers, the latter being merely a highly accurate watch. It is evident that as a spiral spring uncoils, its tension will gradually diminish, and to counteract this the well-known arrangement of the barrel and fusee was adopted which is still continued in the best forms of watches where accuracy is essential, but in the commoner forms where only tolerable accuracy from day to day is required, a long spring is used only, a small part of which comes into play, and the error introduced is not great. Another point of difference between a watch and a clock is in the escapement, inasmuch as the arc of vibration in the former case frequently reaches 270° , while in the latter it seldom exceeds 4° or 5° . Different temperatures cause variations in the time of chronometers as well as in clocks, and this is provided for by the compensating balance wheel which consists of strips of two different metals, the more expansible on the outside, which, when the temperature rises, bends inward bringing the two masses of their extremities nearer the centre of gravity and thus counteracting the effect of the expansion of the wheel as a whole.

Now we frequently require to determine small intervals of time for which a clock or watch, no matter how accurate,

would be unsuitable, and thus special apparatus is required. The commonest form of instrument for this purpose is the stop-watch, which is a modification of the ordinary instrument, provided with a stud by which the mechanism may be started and stopped at given signals. It is usually graduated to fifths of a second and answers very well where great accuracy is not required.

Instruments in which the time is indicated by written marks are known as chronographs and may be constructed in various ways. A simple form of instrument consists of a Morse writing telegraph in circuit with a seconds pendulum. Every time the pendulum passes its lowest point in the arc of vibration it touches a drop of mercury which completes a circuit and causes the style of the instrument to trace a short mark on a strip of paper moved by clock-work, so that the writing consists of a series of dashes separated by equal intervals. A key is also placed in the circuit by means of which the style may be caused to touch the paper at any time independent of the action of the pendulum. In operating the instrument all that is necessary is to count the number of dashes between two signal marks by the key and this gives the number of seconds.

By making use of the isochronous vibrations of a tuning fork we can register time with great accuracy. The ordinary form of tuning-fork chronograph consists of a fork vibrated electrically and carrying a style on one prong which traces a sinuous curve on a moving strip of paper. A second style controlled by a key can be made to trace a mark on this paper at any desired point. If now we wish to measure the time that elapses between two signal marks, all we have to do is to count the number of waves, and when the rate of the fork is known we have the time. An ordinary chronograph of this kind will give indications to one thousandth of a second.

We have now briefly sketched the progress of invention in this particular direction from the sundial to the modern

perfected instruments of precision. We have seen how gradual but sure has been the advance through the various intermediate forms of instruments, each apparently perfect in its day; and we wonder what will be the outcome in future, but no doubt can exist that in this as in all else the path of science will be upward and onward, and future generations will see instruments of precision besides which ours will appear simply as a historical curiosity which served its purpose very well in its day, but has ceased to have any scientific value.
